Zero Point Energy: Planck Radiation Law

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Abstract: The assumption of discrete radiation energy in Planck's 1901 radiation law, conflicted with Planck's belief in radiation of continuous waves. To reconcile his quantum hypothesis with his conception of wave radiation, he avoided the conclusion that radiation energy must be made of particles, and postulated that radiation is a transition between the energy levels of an oscillator. Furthermore, ignoring the symmetry between emission and absorption, he maintained that the absorption of radiation energy is continuous.

Under these assumptions, Planck derived in 1912, a second radiation law in which zero point energy appears.

We show that Planck's derivation of his 1912 radiation law only recovers the Zero Point Energy that he unknowingly assumed in his model from the start.

Furthermore, the distribution law of Planck's 1912 radiation law is, in fact, the approximated Boson Statistics of Planck's 1901 radiation law.

Our main result is that Planck's ZPE radiation law is equivalent to the combined three assumptions of Zero Point Energy

Hypothesis, the Quantum Law, and the approximated Boson Statistics distribution law.

The validity of Planck's 1912 radiation law, and the existence of Planck's Zero Point Energy are doubtful.

Introduction: In 1893, Wien reasoned that the radiation-energy density per unit volume at frequencies between ν, and $\nu+\delta\nu$ of an ideal radiator (black body) is

$$u(\nu,T)d\nu = \frac{\nu^3}{c^4}\Phi(\frac{\nu}{T})d\nu \ .$$

Radiation measurements indicated to Wien that Φ should be of the form

$$\Phi = C_1 e^{-\frac{C_2\nu}{T}} \ ,$$

with some constants C_1, and C_2.

On the other hand,

$$u(\nu,T)d\nu = N_\nu \overline{\varepsilon_\nu} d\nu \ ,$$

where N_ν is the number of radiating oscillators (per unit volume) at frequencies between ν, and $\nu+\delta\nu$, and $\overline{\varepsilon_\nu}$ is the average radiation energy of an oscillator between these frequencies.

Rayleigh computed N_ν as the number of standing waves (per unit volume) in the modes between ν, and $\nu+\delta\nu$,

$$N_\nu d\nu = \frac{8\pi}{c^3}\nu^2 d\nu \ .$$

The radiation energy of the oscillators ε_ν is distributed with Boltzman probability density

$$f(\varepsilon_\nu) = \alpha e^{-\frac{\varepsilon_\nu}{kT}},$$

where α is determined from the condition that the total probability is 1.

Rayleigh assumed that ε_ν may take any value between 0, and ∞.

Then, the Boltzman distribution $f(\varepsilon_\nu)$ is continuous.

The condition

$$\int_{\varepsilon_\nu=0}^{\varepsilon_\nu=\infty} f(\varepsilon_\nu) d\varepsilon_\nu = 1$$

determines

$$\alpha = \frac{1}{kT}.$$

Thus,

$$\overline{\varepsilon_\nu} = \int_{\varepsilon_\nu=0}^{\varepsilon_\nu=\infty} \varepsilon_\nu f(\varepsilon_\nu) d\varepsilon_\nu = \int_{\varepsilon_\nu=0}^{\varepsilon_\nu=\infty} \frac{\varepsilon_\nu}{kT} e^{-\frac{\varepsilon_\nu}{kT}} d\varepsilon_\nu = kT.$$

Consequently, the Rayleigh-Jeans

$$u(\nu, T) = \frac{8\pi}{c^3}\nu^2 kT,$$

disagrees with Wien's law, and with the measurements that $u(\nu, T) \to 0$ for large radiation frequencies.

Wien's approximated law requires that $\overline{\varepsilon_\nu}$ will depend linearly on ν

$$\overline{\varepsilon_\nu} = a\nu,$$

and the failure of the Rayleigh-Jeans argument suggests assuming that ε_ν may take only discrete values $h\nu$, $2h\nu$, $3h\nu$,…That is,

$$\varepsilon_{\nu,n} = nh\nu \,.$$

Then, the Boltzman distribution $f(\varepsilon_{\nu,n})$ is discrete,

$$f(\varepsilon_{\nu,n}) = \alpha e^{-n\frac{h\nu}{kT}} \,.$$

The condition

$$\sum_{n=1}^{\infty} f(\varepsilon_{\nu,n}) = 1$$

determines

$$\alpha = 1 - e^{-\frac{h\nu}{kT}} \,.$$

Thus,

$$\overline{\varepsilon_{\nu}} = \sum_{n=1}^{\infty} \varepsilon_{\nu,n} f(\varepsilon_{\nu,n}) = (1 - e^{-\frac{h\nu}{kT}}) \sum_{n=1}^{\infty} nh\nu e^{-\frac{nh\nu}{kT}} \,.$$

The series $\sum_{n=1}^{\infty} e^{-n\frac{h\nu}{kT}}$ converges uniformly to $\dfrac{1}{1 - \mathrm{e}^{-\frac{h\nu}{kT}}}$, and can be differentiated term by term with respect to $\xi \equiv \dfrac{1}{kT}$. That is,

$$\sum_{n=1}^{\infty} nh\nu e^{-nh\nu\xi} = -\frac{d}{d\xi} \sum_{n=1}^{\infty} e^{-nh\nu\xi} = -\frac{d}{d\xi} \frac{1}{1 - e^{-h\nu\xi}} = \frac{e^{-h\nu\xi} h\nu}{(1 - e^{-h\nu\xi})^2} \,.$$

Therefore,

$$\overline{\varepsilon_{\nu}} = (1 - e^{-h\nu\xi}) \frac{e^{-h\nu\xi} h\nu}{(1 - e^{-h\nu\xi})^2} = \frac{h\nu}{e^{h\nu\frac{1}{kT}} - 1} \,.$$

Then, Planck's 1901 radiation law

$$u(\nu, T) = \frac{8\pi}{c^3} \nu^3 \frac{h}{e^{\frac{h\nu}{kT}} - 1}$$

fits the measurements better than Wien's law.

The assumption of discrete radiation energy, conflicted with Planck's belief in radiation of continuous waves. To reconcile his quantum hypothesis with his conception of wave radiation, he avoided the conclusion that radiation energy must be made of particles, and postulated that radiation is a transition between the energy levels of an oscillator. Furthermore, ignoring the symmetry between emission and absorption, he maintained that the absorption of radiation energy is continuous.

Under these assumptions, Planck derived a second radiation law in which zero point energy appears. We proceed to examine Planck's derivation of his 1912 radiation law.

How Planck obtained Zero point energy: Planck's 1912 oscillator model assumes a probability p_ω for the oscillator to radiate, and a probability $q_\omega = 1 - p_\omega$ to not radiate. He assumes

energy of $\frac{1}{2}\hbar\omega$ with probability $p_{\omega,1}$,

energy of $\frac{3}{2}\hbar\omega$ with probability $p_{\omega,2} = p_{\omega,1}q_\omega$,

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energy of $\frac{2n-1}{2}\hbar\omega$ with probability $p_{\omega,n} = p_{\omega,1}q_\omega^{n-1}$,

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Since $1 = p_{\omega,1} + p_{\omega,2} + ... = p_{\omega,1}(1 + q_\omega + q_\omega^2 + ...) = \dfrac{p_{\omega,1}}{1 - q_\omega} = \dfrac{p_{\omega,1}}{p_\omega}$,

we have

$$p_{\omega,1} = p_\omega, \quad \text{and} \quad p_{\omega,n} = p_\omega q_\omega^{n-1}.$$

Therefore, the oscillator's average entropy is

$$\overline{s_\omega} = -k\left(p_{\omega,1}\ln p_{\omega,1} + p_{\omega,2}\ln p_{\omega,2} + \ldots\right)$$

$$= -kp_\omega\left(\ln p_\omega + q_\omega \ln p_\omega q_\omega + q_\omega^2 \ln p_\omega q_\omega^2 + \ldots\right)$$

$$= -kp_\omega \ln p_\omega\left(1 + q_\omega + q_\omega^2 + \ldots\right) - (kp_\omega q_\omega \ln q_\omega)\left(1 + 2q_\omega + 3q_\omega^2 + \ldots\right)$$

$$= -(kp_\omega \ln p_\omega)\frac{1}{1-q_\omega} - (kp_\omega q_\omega \ln q_\omega)\frac{d}{dq_\omega}\left(1 + q_\omega + q_\omega^2 + \ldots\right)$$

$$= -(kp_\omega \ln p_\omega)\frac{1}{1-q_\omega} - (kp_\omega q_\omega \ln q_\omega)\frac{d}{dq_\omega}\frac{1}{1-q_\omega}$$

$$= -(kp_\omega \ln p_\omega)\frac{1}{1-q_\omega} - (kp_\omega q_\omega \ln q_\omega)\frac{1}{(1-q_\omega)^2}$$

$$= -k\ln p_\omega - k\frac{1-p_\omega}{p_\omega}\ln(1-p_\omega)$$

$$= k\left\{-\ln p_\omega - \frac{1-p_\omega}{p_\omega}\ln\frac{1-p_\omega}{p_\omega} - \frac{1-p_\omega}{p_\omega}\ln p_\omega\right\}$$

$$= k\left\{\frac{1}{p_\omega}\ln\frac{1}{p_\omega} - \left(\frac{1}{p_\omega}-1\right)\ln\left(\frac{1}{p_\omega}-1\right)\right\}, \qquad (1)$$

The average radiation energy of an oscillator is

$$\overline{\varepsilon_\omega} = \frac{1}{2}\hbar\omega p_{\omega,1} + \frac{3}{2}\hbar\omega p_{\omega,2} + \frac{5}{2}\hbar\omega p_{\omega,3} + \frac{7}{2}\hbar\omega p_{\omega,4} + \ldots$$

$$= \frac{1}{2}\hbar\omega\, p_\omega\left\{1 + 3q_\omega + 5q_\omega^2 + 7q_\omega^3 + \ldots\right\}$$

$$= \frac{1}{2}\hbar\omega\, p_\omega\left\{\left(1 + q_\omega + q_\omega^2 + q_\omega^3\ldots\right) + 2q_\omega\left(1 + 2q_\omega + 3q_\omega^2\ldots\right)\right\}$$

$$= \frac{1}{2}\hbar\omega\, p_\omega\left\{\frac{1}{1-q_\omega} + 2q_\omega\frac{d}{dq_\omega}\left(1 + q_\omega + q_\omega^2 + q_\omega^3\ldots\right)\right\}$$

$$= \frac{1}{2}\hbar\omega\, p_\omega \left\{ \frac{1}{p_\omega} + 2q_\omega \frac{d}{dq}\frac{1}{1-q_\omega} \right\}$$

$$= \frac{1}{2}\hbar\omega\, p_\omega \left\{ \frac{1}{p_\omega} + 2q_\omega \frac{1}{(1-q_\omega)^2} \right\}$$

$$= \hbar\omega \left\{ \frac{1}{2} + (1-p_\omega)\frac{1}{p_\omega} \right\}$$

$$= \hbar\omega \left(\frac{1}{p_\omega} - \frac{1}{2} \right). \qquad (2)$$

Therefore, $\overline{s_\omega}$ in terms of $\overline{\varepsilon_\omega}$ is

$$\overline{s_\omega} = k \left\{ \left(\frac{\overline{\varepsilon_\omega}}{\hbar\omega} + \frac{1}{2} \right) \ln \left(\frac{\overline{\varepsilon_\omega}}{\hbar\omega} + \frac{1}{2} \right) - \left(\frac{\overline{\varepsilon_\omega}}{\hbar\omega} - \frac{1}{2} \right) \ln \left(\frac{\overline{\varepsilon_\omega}}{\hbar\omega} - \frac{1}{2} \right) \right\}. \quad (3)$$

Finally, using $\frac{\partial \overline{s_\omega}}{\partial \overline{\varepsilon_\omega}} = \frac{1}{T}$, we obtain Planck's 1912 radiation law:

$$\overline{\varepsilon_\omega} = \frac{\hbar\omega}{e^{\frac{\hbar\omega}{kT}} - 1} + \frac{1}{2}\hbar\omega. \qquad (4)$$

The Zero Point Energy of $\frac{1}{2}\hbar\omega$ is the mid-energy that was assumed with probability p_ω. Planck's derivation only recovers the ZPE that he assumed at the start.

Characterization of Planck's 1912 radiation law. The distribution law of Planck's 1912 radiation law is, in fact, the approximated Boson Statistics of Planck's 1901 radiation law. Planck's 1901 model [ref. 3], can be reworked to obtain his 1912 law, provided that the approximated Boson statistics is assumed.

Our main result is that Planck's ZPE radiation law is equivalent to the combined assumptions of Zero point energy Hypothesis, the quantum law, and the approximated Boson Statistics distribution law.

We prove:

The Radiation Law of equation (4)

$$\overline{\varepsilon_\omega} = \frac{\hbar\omega}{e^{\frac{\hbar\omega}{kT}} - 1} + \frac{1}{2}\hbar\omega$$

is equivalent to the three combined assumptions per mode in $[\omega, \omega + d\omega]$

- ***ZPE Hypothesis****: Each radiator has zero point energy* $\frac{1}{2}\hbar\omega$,
- ***Quantum Law****: Energy is radiated in multiples of* $\varepsilon_\omega = \hbar\omega$**,**
- ***Approximated Bosons Statistics****:* P *quanta can be distributed among* N *radiators in approximately* $W_N = \frac{(P+N-1)!}{P!(N-1)!}$ *ways.*

($\Rightarrow$)

Equation (4) mandates the quantum radiation law $\varepsilon_\omega = \hbar\omega$**,** because the negation of the quantum radiation law implies the negation of equation (4).

Equation (4) implies zero point energy of $\hbar\omega / 2$, since for $T \to 0$, $\overline{\varepsilon_\omega} \to \hbar\omega / 2$.

We want to show that (4) implies the approximated bosons statistics assumption. From

$$\overline{\varepsilon_\omega} = \frac{\hbar\omega}{e^{\frac{\hbar\omega}{kT}} - 1} + \frac{1}{2}\hbar\omega = \frac{\varepsilon_\omega}{2}\frac{e^{\frac{\varepsilon_\omega}{kT}} + 1}{e^{\frac{\varepsilon_\omega}{kT}} - 1}, \quad (5)$$

we have

$$\frac{1}{T} = \frac{k}{\varepsilon_\omega}\left\{\ln\left(\frac{\overline{\varepsilon_\omega}}{\varepsilon_\omega} + \frac{1}{2}\right) - \ln\left(\frac{\overline{\varepsilon_\omega}}{\varepsilon_\omega} - \frac{1}{2}\right)\right\}. \quad (6)$$

That is,

$$\frac{\partial \overline{s_\omega}}{\partial \overline{\varepsilon_\omega}} = \frac{k}{\varepsilon_\omega}\left\{\ln\left(\frac{\overline{\varepsilon_\omega}}{\varepsilon_\omega} + \frac{1}{2}\right) - \ln\left(\frac{\overline{\varepsilon_\omega}}{\varepsilon_\omega} - \frac{1}{2}\right)\right\}. \quad (7)$$

Integrating (7), the average entropy of a radiator per mode in $[\omega, \omega + d\omega]$ is

$$\overline{s_\omega} = k\left\{\left(\frac{\overline{\varepsilon_\omega}}{\varepsilon_\omega} + \frac{1}{2}\right)\ln\left(\frac{\overline{\varepsilon_\omega}}{\varepsilon_\omega} + \frac{1}{2}\right) - \left(\frac{\overline{\varepsilon_\omega}}{\varepsilon_\omega} - \frac{1}{2}\right)\ln\left(\frac{\overline{\varepsilon_\omega}}{\varepsilon_\omega} - \frac{1}{2}\right)\right\}. \quad (8)$$

We assume P_ω energy-quanta $\varepsilon_\omega = \hbar\omega$ distributed between N_ω radiators at frequency ω. Each of the radiators has zero point energy $\frac{1}{2}\varepsilon_\omega = \frac{1}{2}\hbar\omega$, included in his average radiation energy $\overline{\varepsilon_\omega}$. Therefore, the balance of radiation energy at frequency ω is

$$\varepsilon_\omega P_\omega + \frac{1}{2}\varepsilon_\omega N_\omega = N_\omega \overline{\varepsilon_\omega}. \quad (9)$$

Substituting this into (8),

$$\overline{s_\omega} = k\left\{\left(\frac{P_\omega}{N_\omega} + 1\right)\ln\left(\frac{P_\omega}{N_\omega} + 1\right) - \frac{P_\omega}{N_\omega}\ln\frac{P_\omega}{N_\omega}\right\}. \quad (10)$$

Therefore, the total entropy of the N_ω radiators per mode in $[\omega, \omega + d\omega]$ is

$$S_{N_\omega} = N_\omega \overline{s_\omega} = k\left\{(P_\omega + N_\omega)\left(\ln(P_\omega + N_\omega) - \ln N_\omega\right) - P_\omega\left(\ln P_\omega - \ln N_\omega\right)\right\}. \quad (11)$$

Using Sterling's formula $\ln M! \approx M \ln M - M$,

$$S_{N_\omega} \approx k\left\{\ln\left(P_\omega + N_\omega\right)! + P_\omega + N_\omega - \ln N_\omega! - N_\omega - \ln P_\omega! - P_\omega\right\}$$

$$= k \ln \frac{\left(P_\omega + N_\omega\right)!}{P_\omega! N_\omega!}. \quad (12)$$

That is, the P_ω quanta can be distributed among the N_ω radiators in

$$W_{N_\omega} = \frac{\left(P_\omega + N_\omega\right)!}{P_\omega! N_\omega!} \quad (13)$$

ways.

As Planck comments [ref.3], equation (13) approximates well the formula $W_N = \frac{\left(P + N - 1\right)!}{P!\left(N-1\right)!}$ of the bosons statistics.

$(\Leftarrow)$

Conversely, assume P_ω quanta of energy $\varepsilon_\omega = \hbar\omega$, that are distributed between N_ω radiators at frequency ω in

$$W_{N_\omega} = \frac{\left(P_\omega + N_\omega\right)!}{N_\omega! P_\omega!}$$

ways. The average entropy of a radiator per mode in $[\omega, \omega + d\omega]$ is

$$\overline{s_\omega} = \frac{1}{N_\omega} k \ln \frac{\left(P_\omega + N_\omega\right)!}{N_\omega! P_\omega!} = \frac{1}{N_\omega} k\left\{\ln\left(P_\omega + N_\omega\right)! - \ln N_\omega! - \ln P_\omega!\right\}$$

Using Sterling's formula $\ln M! \approx M \ln M - M$,

$$\approx \frac{1}{N_\omega} k\left\{(P_\omega + N_\omega)\ln(P_\omega + N_\omega) - N_\omega \ln N_\omega - P_\omega \ln P_\omega\right\}$$

$$= \frac{1}{N_\omega} k\left\{(P_\omega + N_\omega)\left(\ln(P_\omega + N_\omega) - \ln N_\omega\right) - P_\omega\left(\ln P_\omega - \ln N_\omega\right)\right\}$$

$$= k\left\{\left(\frac{P_\omega}{N_\omega}+1\right)\ln\left(\frac{P_\omega}{N_\omega}+1\right)-\frac{P_\omega}{N_\omega}\ln\frac{P_\omega}{N_\omega}\right\},$$

which is equation (10).

Assuming zero point energy $\frac{1}{2}\varepsilon_\omega = \frac{1}{2}\hbar\omega$ for each of the radiators, we obtain the balance equation (9)

$$\varepsilon_\omega P_\omega + \frac{1}{2}\varepsilon_\omega N_\omega = N_\omega \overline{\varepsilon_\omega}.$$

Plugging (9) into (10), we obtain equation (8). Differentiating (8) with respect to $\overline{\varepsilon_\omega}$ we get (7), from which we conclude (6), and (5), which is Planck's radiation law with ZPE.

Doubts over ZPE in the Radiation Law. Comparing equations (2) and (4), we have

$$\frac{1}{p_\omega}-\frac{1}{2}=\frac{1}{e^{\frac{\hbar\omega}{kT}}-1}+\frac{1}{2}.$$

That is,

$$p_\omega = 1-e^{-\frac{\hbar\omega}{kT}}.$$

Consequently, at large frequencies Planck's ZPE is assumed to be present almost certainly.

This casts uncertainty on the validity of Planck's Zero Point Energy.

The energy density of Planck's 1901 radiation law yields total energy density

$$u(T) = \int_{\nu=0}^{\nu=\infty} u(\nu,T)d\nu = \frac{8\pi}{c^3}\int_{\nu=0}^{\nu=\infty} \nu^2 \frac{h\nu}{e^{\frac{h\nu}{kT}}-1}d\nu = \frac{8\pi^5}{15(hc)^3}(kT)^4,$$

which is Stephan-Boltzman radiation law.

The addition of Zero Point Energy of $\frac{1}{2}h\nu$ in Planck's 1912 radiation law, adds to the total energy density

$$\frac{8\pi}{c^3}\int_{\nu=0}^{\nu=\infty} \nu^2 (\frac{1}{2}h\nu)d\nu = \infty.$$

This consequence of Planck's 1912 radiation law has been given names such as "photon self-energy", "vacuum polarization", "vacuum fluctuations", and "mass renormalization", but no-one understands what unobservable infinite zero point energy means. Consequently, the validity of Planck's 1912 radiation law, and the existence of Planck's Zero Point Energy are doubtful.

REFERENCES

1. Planck, M. . {Annalen der physik 37 (1912):p.642}.

2. Planck, M. *The Theory of Heat Radiation*, Dover 1959.

3. Planck, M. "*On the theory of the law of the distribution of energy in the normal spectrum*" {Annalen der physik 4 (1901):p.553}.

4. *Planck's Original Papers in Quantum Physics*, Wiley, 1972.

5. Planck, M. {Verh.d. Deutsch. Physikal Gesellschaft, 2 (1900);p.202}. #15 in *Early Concepts of Energy in Atomic Physics*, Dowden, 1979