The online version of this document is available at http://tutorial.math.lamar.edu. At the above web site you will find not only the online version of this document but also pdf versions of each section, chapter and complete set of notes.

Preface

Here are my online notes for my Calculus I course that I teach here at Lamar University. Despite the fact that these are my "class notes" they should be accessible to anyone wanting to learn Calculus I or needing a refresher in some of the early topics in calculus.

I've tried to make these notes as self contained as possible and so all the information needed to read through them is either from an Algebra or Trig class or contained in other sections of the notes.

Here are a couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn calculus I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn't covered in class.
2. Because I want these notes to provide some more examples for you to read through, I don't always work the same problems in class as those given in the notes. Likewise, even if I do work some of the problems in here I may work fewer problems in class than are presented here.
3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible in writing these up, but the reality is that I can't anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I've not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.
4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.

Review

Introduction

Technically a student coming into a Calculus class is supposed to know both Algebra and Trigonometry. The reality is often much different however. Most students enter a Calculus class woefully unprepared for both the algebra and the trig that is in a Calculus class. This is very unfortunate since good algebra skills are absolutely vital to successfully completing any Calculus course and if your Calculus course includes trig (as this one does) good trig skills are also important in many sections.

The intent of this chapter is to do a very cursory review of some algebra and trig skills that are absolutely vital to a calculus course. This chapter is not inclusive in the algebra and trig skills that are needed to be successful in a Calculus course. It only includes those topics that most students are particularly deficient in. For instance factoring is also vital to completing a standard calculus class but is not included here. For a more in depth review you should visit my Algebra/Trig review or my full set of Algebra notes at http://tutorial.math.lamar.edu.

Note that even though these topics are very important to a Calculus class I rarely cover all of these in the actual class itself. We simply don't have the time to do that. I do cover certain portions of this chapter in class, but for the most part I leave it to the students to read this chapter on their own.

Here is a list of topics that are in this chapter. I've also denoted the sections that I typically cover during the first couple of days of a Calculus class.

Review : Functions – Here is a quick review of functions, function notation and a couple of fairly important ideas about functions.

Review : Inverse Functions – A quick review of inverse functions and the notation for inverse functions.

Review : Trig Functions – A review of trig functions, evaluation of trig functions and the unit circle. This section usually gets a quick review in my class.

Review : Solving Trig Equations – A reminder on how to solve trig equations. This section is always covered in my class.

Review : Exponential Functions – A review of exponential functions. This section usually gets a quick review in my class.

Review : Logarithm Functions – A review of logarithm functions and logarithm properties. This section usually gets a quick review in my class.

Review : Exponential and Logarithm Equations – How to solve exponential and logarithm equations. This section is always covered in my class.

Review : Common Graphs – This section isn't much. It's mostly a collection of graphs of many of the common functions that are liable to be seen in a Calculus class.

Review : Functions

In this section we're going to make sure that you're familiar with functions and function notation. Functions and function notation will appear in almost every section in a Calculus class and so you will need to be able to deal with them.

First, what exactly is a function? An equation will be a function if for any x in the domain of the equation (the domain is all the x's that can be plugged into the equation) the equation will yield exactly one value of y.

This is usually easier to understand with an example.

Example 1 Determine if each of the following are functions.

(a) $y = x^2 + 1$

(b) $y^2 = x + 1$

Solution

(a) This first one is a function. Given an x there is only one way to square it and so no matter what value of x you put into the equation there is only one possible value of y.

(b) The only difference between this equation and the first is that we moved the exponent off the x and onto the y. This small change is all that is required, in this case, to change the equation from a function to something that isn't a function.

To see that this isn't a function is fairly simple. Choose a value of x, say $x=3$ and plug this into the equation.

$$y^2 = 3 + 1 = 4$$

Now, there are two possible values of y that we could use here. We could use $y = 2$ or $y = -2$. Since there are two possible values of y that we get from a single x this equation isn't a function.

Note that this only needs to be the case for a single value of x to make an equation not be a function. For instance we could have used $x=-1$ and in this case we would get a single y ($y=0$). However, because of what happens at $x=3$ this equation will not be a function.

Next we need to take a quick look at function notation. Function notation is nothing more than a fancy way of writing the y in a function that will allow us to simplify notation and some of our work a little.

Let's take a look at the following function.

$$y = 2x^2 - 5x + 3$$

Using function notation we can write this as any of the following.

$$f(x) = 2x^2 - 5x + 3$$
$$g(x) = 2x^2 - 5x + 3$$
$$h(x) = 2x^2 - 5x + 3$$
$$R(x) = 2x^2 - 5x + 3$$
$$w(x) = 2x^2 - 5x + 3$$
$$y(x) = 2x^2 - 5x + 3$$
$$\vdots$$

Recall that this is NOT a letter times *x*, this is just a fancy way of writing *y*.

So, why is this useful? Well let's take the function above and let's get the value of the function at $x=-3$. Using function notation we represent the value of the function at $x=-3$ as $f(-3)$. Function notation gives us a nice compact way of representing function values.

Now, how do we actually evaluate the function? That's really simple. Everywhere we see an *x* on the right side we will substitute whatever is in the parenthesis on the left side. For our function this gives,

$$\begin{aligned} f(-3) &= 2(-3)^2 - 5(-3) + 3 \\ &= 2(9) + 15 + 3 \\ &= 36 \end{aligned}$$

Let's take a look at some more function evaluation.

Example 2 Given $f(x) = -x^2 + 6x - 11$ find each of the following.

(a) $f(2)$

(b) $f(-10)$

(c) $f(t)$

(d) $f(t-3)$

(e) $f(x-3)$

(f) $f(4x-1)$

Solution

(a) $f(2) = -(2)^2 + 6(2) - 11 = -3$

(b) $f(-10) = -(-10)^2 + 6(-10) - 11 = -100 - 60 - 11 = -171$

Be careful when squaring negative numbers!

(c) $f(t) = -t^2 + 6t - 11$

Remember that we substitute for the *x*'s WHATEVER is in the parenthesis on the left. Often this will be something other than a number. So, in this case we put *t*'s in for all the *x*'s on the left.

(d) $f(t-3) = -(t-3)^2 + 6(t-3) - 11 = -t^2 + 12t - 38$

Often instead of evaluating functions at numbers or single letters we will have some fairly complex evaluations so make sure that you can do these kinds of evaluations.

(e) $f(x-3) = -(x-3)^2 + 6(x-3) - 11 = -x^2 + 12x - 38$

The only difference between this one and the previous one is that I changed the *t* to an *x*. Other than that there is absolutely no difference between the two! Don't get excited if an *x* appears inside the parenthesis on the left.

(f) $f(4x-1) = -(4x-1)^2 + 6(4x-1) - 11 = -16x^2 + 32x - 18$

This one is not much different from the previous part. All we did was change the equation that we were plugging into function.

All throughout a calculus course we will be finding roots of functions. A root of a function is nothing more than a number for which the function is zero. In other words, finding the roots of a function, *g(x)*, is equivalent to solving

$$g(x) = 0$$

Example 3 Determine all the roots of $f(t) = 9t^3 - 18t^2 + 6t$

Solution

So we will need to solve,

$$9t^3 - 18t^2 + 6t = 0$$

First, we should factor the equation as much as possible. Doing this gives,

$$3t\left(3t^2 - 6t + 2\right) = 0$$

Next recall that if a product of two things are zero then one (or both) of them had to be zero. This means that,

$$3t = 0 \qquad \text{OR,}$$

$$3t^2 - 6t + 2 = 0$$

From the first it's clear that one of the roots must then be *t=0*. To get the remaining roots we will need to use the quadratic formula on the second equation. Doing this gives,

$$\begin{aligned} t &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(3)(2)}}{2(3)} \\ &= \frac{6 \pm \sqrt{12}}{6} \\ &= \frac{6 \pm \sqrt{(4)(3)}}{6} \\ &= \frac{6 \pm 2\sqrt{3}}{6} \\ &= \frac{3 \pm \sqrt{3}}{3} \\ &= 1 \pm \frac{1}{3}\sqrt{3} \\ &= 1 \pm \frac{1}{\sqrt{3}} \end{aligned}$$

In order to remind you how to simplify radicals we gave several forms of the answer.

To complete the problem, here is a complete list of all the roots of this function.

$$t = 0,\ t = \frac{3+\sqrt{3}}{3},\ \ t = \frac{3-\sqrt{3}}{3}$$

Note we didn't use the final form for the roots from the quadratic. This is usually where we'll stop with the simplification for these kinds of roots. Also note that, for the sake of the practice, we broke up the compact form for the two roots of the quadratic. You will need to be able to do this so make sure that you can.

This example had a couple of points other than finding roots of functions.

The first was to remind you of the quadratic formula. This won't be the first time that you'll need it in this class.

The second was to get you used to seeing "messy" answers. In fact, the answers in the above list are not that messy. However, most students come out of an Algebra class very used to seeing only integers and the occasional "nice" fraction as answers.

So, here is fair warning. In this class I often will intentionally make the answers look "messy" just to get you out of the habit of always expecting "nice" answers. In "real life" (whatever that is) the answer is rarely a simple integer such as two. In most problems the answer will be a decimal that came about from a messy fraction and/or an answer that involved radicals.

The next topic that we need to discuss here is that of **function composition**. The composition of *f(x)* and *g(x)* is

$$\left(f \circ g\right)\left(x\right) = f\left(g\left(x\right)\right)$$

In other words, compositions are evaluated by plugging the second function listed into the first function listed. Note as well that order is important here. Interchanging the order will usually result in a different answer.

Example 4 Given $f\left(x\right) = 3x^2 - x + 10$ and $g\left(x\right) = 1 - 20x$ find each of the following.

(a) $\left(f \circ g\right)\left(5\right)$

(b) $\left(f \circ g\right)\left(x\right)$

(c) $\left(g \circ f\right)\left(x\right)$

(d) $\left(g \circ g\right)\left(x\right)$

Solution

(a) In this case we've got a number instead of an *x* but it works in exactly the same way.

$$\begin{aligned}\left(f \circ g\right)\left(5\right) &= f\left(g\left(5\right)\right)\\ &= f\left(-99\right)\\ &= 29512\end{aligned}$$

(b)

$$\begin{aligned}\left(f \circ g\right)\left(x\right) &= f\left(g\left(x\right)\right)\\ &= f\left(1 - 20x\right)\\ &= 3\left(1 - 20x\right)^2 - \left(1 - 20x\right) + 10\\ &= 3\left(1 - 40x + 400x^2\right) - 1 + 20x + 10\\ &= 1200x^2 - 100x + 12\end{aligned}$$

Compare this answer to the next part and notice that answers are NOT the same. The order in which the functions are listed is important!

(c)

$$\begin{aligned}\left(g \circ f\right)\left(x\right) &= g\left(f\left(x\right)\right)\\ &= g\left(3x^2 - x + 10\right)\\ &= 1 - 20\left(3x^2 - x + 10\right)\\ &= -60x^2 + 20x - 199\end{aligned}$$

And just to make the point. This answer is different from the previous part. Order is important in composition.

(d) In this case do not get excited about the fact that it's the same function. Composition still works the same way.

$$\begin{aligned}(g \circ g)(x) &= g(g(x)) \\ &= g(1-20x) \\ &= 1-20(1-20x) \\ &= 400x-19\end{aligned}$$

Let's work one more example that will lead us into the next section.

Example 5 Given $f(x) = 3x-2$ and $g(x) = \frac{1}{3}x + \frac{2}{3}$ find each of the following.

(a) $(f \circ g)(x)$

(b) $(g \circ f)(x)$

Solution

(a)

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f\left(\frac{1}{3}x + \frac{2}{3}\right) \\ &= 3\left(\frac{1}{3}x + \frac{2}{3}\right) - 2 \\ &= x + 2 - 2 \\ &= x\end{aligned}$$

(b)

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(3x-2) \\ &= \frac{1}{3}(3x-2) + \frac{2}{3} \\ &= x - \frac{2}{3} + \frac{2}{3} \\ &= x\end{aligned}$$

In this case the two compositions where the same and in fact the answer was very simple.

$$(f \circ g)(x) = (g \circ f)(x) = x$$

This will usually not happen. However, when the two compositions are the same, or more specifically when the two compositions are both *x* there is a very nice relationship between the two functions. We will take a look at that relationship in the next section.

Review : Inverse Functions

In the last example from the previous section we looked at the two functions $f(x) = 3x - 2$ and $g(x) = \frac{x}{3} + \frac{2}{3}$ and saw that

$$(f \circ g)(x) = (g \circ f)(x) = x$$

and as noted in that section this means that there is a nice relationship between these two functions. Let's see just what that relationship is. Consider the following evaluations.

$$f(-1) = 3(-1) - 2 = -5 \qquad \Rightarrow \qquad g(-5) = \frac{-5}{3} + \frac{2}{3} = \frac{-3}{3} = -1$$

$$g(2) = \frac{2}{3} + \frac{2}{3} = \frac{4}{3} \qquad \Rightarrow \qquad f\left(\frac{4}{3}\right) = 3\left(\frac{4}{3}\right) - 2 = 4 - 2 = 2$$

In the first case we plugged $x = -1$ into $f(x)$ and got a value of -5. We then turned around and plugged $x = -5$ into $g(x)$ and got a value of -1, the number that we started off with.

In the second case we did something similar. Here we plugged $x = 2$ into $g(x)$ and got a value of $\frac{4}{3}$, we turned around and plugged this into $f(x)$ and got a value of 2, which is again the number that we started with.

Note that we really are doing some function composition here. The first case is really,

$$(g \circ f)(-1) = g\left[f(-1)\right] = g\left[-5\right] = -1$$

and the second case is really,

$$(f \circ g)(2) = f\left[g(2)\right] = f\left[\frac{4}{3}\right] = 2$$

Note as well that these both agree with the formula for the compositions that we found in the previous section. We get back out of the function evaluation the number that we originally plugged into the composition.

So, just what is going on here? In some way we can think of these two functions as undoing what the other did to a number. In the first case we plugged $x = -1$ into $f(x)$ and then plugged the result from this function evaluation back into $g(x)$ and in some way $g(x)$ undid what $f(x)$ had done to $x = -1$ and gave us back the original x that we started with.

Function pairs that exhibit this behavior are called **inverse functions**. Before formally defining inverse functions and the notation that we're going to use for them we need to get a definition out of the way.

A function is called **one-to-one** if no two values of *x* produce the same *y*. Mathematically this is the same as saying,

$$f(x_1) \neq f(x_2) \qquad \text{whenever} \qquad x_1 \neq x_2$$

So, a function is one-to-one if whenever we plug different values into the function we get different function values.

Some times it is easier to understand this definition if we see a function that isn't one-to-one. Let's take a look at a function that isn't one-to-one. The function $f(x) = x^2$ is not one-to-one because both $f(-2) = 4$ and $f(2) = 4$. In other words there are two different values of *x* that produce the same value of *y*. Note that we can turn $f(x) = x^2$ into a one-to-one function if we restrict ourselves to $0 \leq x < \infty$. This can sometimes be done with functions.

Showing that a function is one-to-one is often tedious and/or difficult. For the most part we are going to assume that the functions that we're going to be dealing with in this course are either one-to-one or we have restricted the domain of the function to get it to be a one-to-one function.

Now, let's formally define just what inverse functions are. Given two one-to-one functions $f(x)$ and $g(x)$ if

$$(f \circ g)(x) = x \qquad \text{AND} \qquad (g \circ f)(x) = x$$

then we say that $f(x)$ and $g(x)$ are **inverses** of each other. More specifically we will say that $g(x)$ is the **inverse** of $f(x)$ and denote it by

$$g(x) = f^{-1}(x)$$

Likewise we could also say that $f(x)$ is the **inverse** of $g(x)$ and denote it by

$$f(x) = g^{-1}(x)$$

The notation that we use really depends upon the problem. In most cases either is acceptable.

For the two functions that we started off this section with we could write either of the following two sets of notation.

$$f(x) = 3x - 2 \qquad f^{-1}(x) = \frac{x}{3} + \frac{2}{3}$$

$$g(x) = \frac{x}{3} + \frac{2}{3} \qquad g^{-1}(x) = 3x - 2$$

Now, be careful with the notation for inverses. The "-1" is NOT an exponent despite the fact that is sure does look like one! When dealing with inverse functions we've got to remember that

$$f^{-1}(x) \neq \frac{1}{f(x)}$$

This is one of the more common mistakes that students make when first studying inverse functions.

The process for finding the inverse of a function is a fairly simple one although there are a couple of steps that can on occasion be somewhat messy. Here is the process

Finding the Inverse of a Function

Given the function $f(x)$ we want to find the inverse function, $f^{-1}(x)$.

1. First, replace $f(x)$ with *y*. This is done to make the rest of the process easier.
2. Replace every *x* with a *y* and replace every *y* with an *x*.
3. Solve the equation from Step 2 for *y*. This is the step where mistakes are most often made so be careful with this step.
4. Replace *y* with $f^{-1}(x)$. In other words, we've managed to find the inverse at this point!
5. Verify your work by checking that $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$ are both true. This work can sometimes be messy making it easy to make mistakes so again be careful.

That's the process. Most of the steps are not all that bad but as mentioned in the process there are a couple of steps that we really need to be careful with since it is easy to make mistakes in those steps.

In the verification step we technically really do need to check that both $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$ are true. For all the functions that we are going to be looking at in this course if one is true then the other will also be true. However, there are functions (they are beyond the scope of this course however) for which it is possible for only of these to be true. This is brought up because in all the problems here we will be just checking one of them. We just need to always remember that technically we should check both.

Let's work some examples.

Example 1 Given $f(x) = 3x - 2$ find $f^{-1}(x)$.

Solution

Now, we already know what the inverse to this function is as we've already done some work with it. However, it would be nice to actually start with this since we know what we should get. This will work as a nice verification of the process.

So, let's get started. We'll first replace $f(x)$ with *y*.

$$y = 3x - 2$$

Next, replace all *x*'s with *y* and all *y's* with *x*.

$$x = 3y - 2$$

Now, solve for *y*.

$$\begin{aligned} x + 2 &= 3y \\ \frac{1}{3}(x+2) &= y \\ \frac{x}{3} + \frac{2}{3} &= y \end{aligned}$$

Finally replace *y* with $f^{-1}(x)$.

$$f^{-1}(x) = \frac{x}{3} + \frac{2}{3}$$

Now, we need to verify the results. We already took care of this in the previous section, however, we really should follow the process so we'll do that here. It doesn't matter which of the two that we check we just need to check one of them. This time we'll check that $(f \circ f^{-1})(x) = x$ is true.

$$\begin{aligned} (f \circ f^{-1})(x) &= f\left[f^{-1}(x)\right] \\ &= f\left[\frac{x}{3} + \frac{2}{3}\right] \\ &= 3\left(\frac{x}{3} + \frac{2}{3}\right) - 2 \\ &= x + 2 - 2 \\ &= x \end{aligned}$$

Example 2 Given $g(x) = \sqrt{x-3}$ find $g^{-1}(x)$.

Solution

The fact that we're using $g(x)$ instead of $f(x)$ doesn't change how the process works. Here are the first few steps.

$$y = \sqrt{x-3}$$
$$x = \sqrt{y-3}$$

Now, to solve for y we will need to first square both sides and then proceed as normal.

$$\begin{aligned} x &= \sqrt{y-3} \\ x^2 &= y-3 \\ x^2+3 &= y \end{aligned}$$

This inverse is then,

$$g^{-1}(x) = x^2 + 3$$

Finally let's verify and this time we'll use the other one just so we can say that we've gotten both down somewhere in an example.

$$\begin{aligned} \left(g^{-1} \circ g\right)(x) &= g^{-1}\left[g(x)\right] \\ &= g^{-1}\left(\sqrt{x-3}\right) \\ &= \left(\sqrt{x-3}\right)^2 + 3 \\ &= x-3+3 \\ &= x \end{aligned}$$

So, we did the work correctly and we do indeed have the inverse.

The next example can be a little messy so be careful with the work here.

Example 3 Given $h(x) = \dfrac{x+4}{2x-5}$ find $h^{-1}(x)$.

Solution

The first couple of steps are pretty much the same as the previous examples so here they are,

$$y = \frac{x+4}{2x-5}$$
$$x = \frac{y+4}{2y-5}$$

Now, be careful with the solution step. With this kind of problem it is very easy to make a mistake here.

$$
\begin{aligned}
x(2y-5) &= y+4 \\
2xy-5x &= y+4 \\
2xy-y &= 4+5x \\
(2x-1)y &= 4+5x \\
y &= \frac{4+5x}{2x-1}
\end{aligned}
$$

So, if we've done all of our work correctly the inverse should be,

$$h^{-1}(x) = \frac{4+5x}{2x-1}$$

Finally we'll need to do the verification. This is also a fairly messy process and it doesn't really matter which one we work with.

$$
\begin{aligned}
(h \circ h^{-1})(x) &= h\left[h^{-1}(x)\right] \\
&= h\left[\frac{4+5x}{2x-1}\right] \\
&= \frac{\frac{4+5x}{2x-1}+4}{2\left(\frac{4+5x}{2x-1}\right)-5}
\end{aligned}
$$

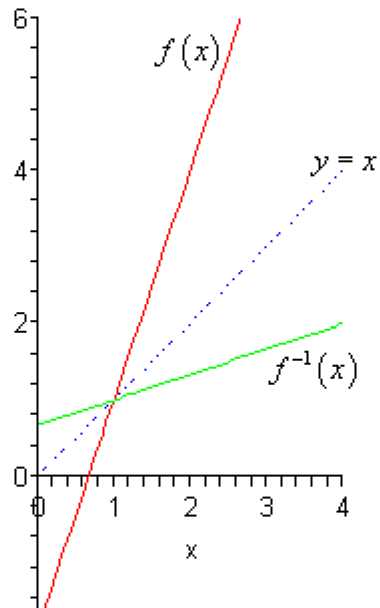
Okay, this is a mess. Let's simplify things up a little bit by multiplying the numerator and denominator by $2x-1$.

$$
\begin{aligned}
(h \circ h^{-1})(x) &= \frac{2x-1}{2x-1}\,\frac{\frac{4+5x}{2x-1}+4}{2\left(\frac{4+5x}{2x-1}\right)-5} \\
&= \frac{(2x-1)\left(\frac{4+5x}{2x-1}+4\right)}{(2x-1)\left(2\left(\frac{4+5x}{2x-1}\right)-5\right)} \\
&= \frac{4+5x+4(2x-1)}{2(4+5x)-5(2x-1)} \\
&= \frac{4+5x+8x-4}{8+10x-10x+5} \\
&= \frac{13x}{13} \\
&= x
\end{aligned}
$$

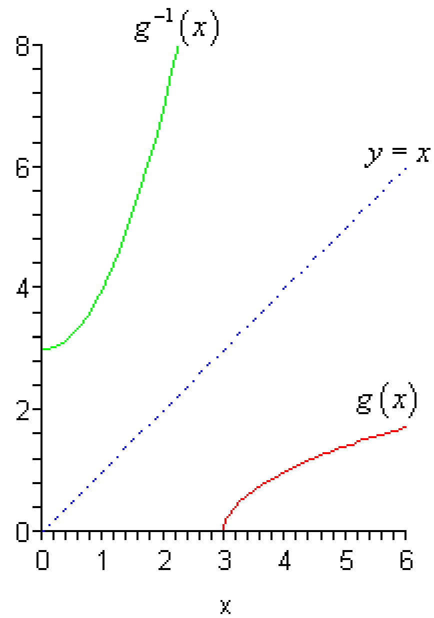
Wow. That was a lot of work, but it all worked out in the end. We did all of our work correctly and we do in fact have the inverse.

There is one final topic that we need to address quickly before we leave this section. There is an interesting relationship between the graph of a function and the graph of its inverse.

Here is the graph of the function and inverse from the first two examples.



Example 1



Example 2

In both cases we can see that the graph of the inverse is a reflection of the actual function about the line $y = x$. This will always be the case with the graphs of a function and its inverse.

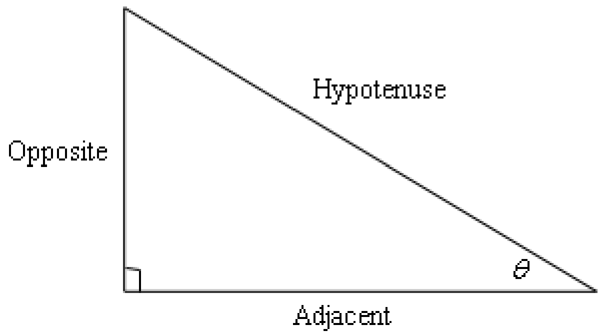
Review : Trig Functions

The intent of this section is to remind you of some of the more important (from a Calculus standpoint…) topics from a trig class. One of the most important (but not the first) of these topics will be how to use the unit circle. We will actually leave the most important topic to the next section.

First let's start with the six trig functions and how they relate to each other.

$$
\begin{aligned}
&\cos(x) && \sin(x) \\
&\tan(x) = \frac{\sin(x)}{\cos(x)} && \cot(x) = \frac{\cos(x)}{\sin(x)} = \frac{1}{\tan(x)} \\
&\sec(x) = \frac{1}{\cos(x)} && \csc(x) = \frac{1}{\sin(x)}
\end{aligned}
$$

Recall as well that all the trig functions can be defined in terms of a right triangle.



From this right triangle we get the following definitions of the six trig functions.

$$
\begin{aligned}
\cos\theta &= \frac{\text{adjacent}}{\text{hypotenuse}} & \sin\theta &= \frac{\text{opposite}}{\text{hypotenuse}} \\
\tan\theta &= \frac{\text{opposite}}{\text{adjacent}} & \cot\theta &= \frac{\text{adjacent}}{\text{opposite}} \\
\sec\theta &= \frac{\text{hypotenuse}}{\text{adjacent}} & \csc\theta &= \frac{\text{hypotenuse}}{\text{opposite}}
\end{aligned}
$$

Remembering both the relationship between all six of the trig functions and their right triangle definitions will be useful in this course on occasion.

Next, we need to touch on radians. In most trig classes instructors tend to concentrate on doing everything in terms of degrees (probably because it's easier to visualize degrees). However, in a calculus course almost everything is done in radians. The following table gives some of the basic angles in both degrees and radians.

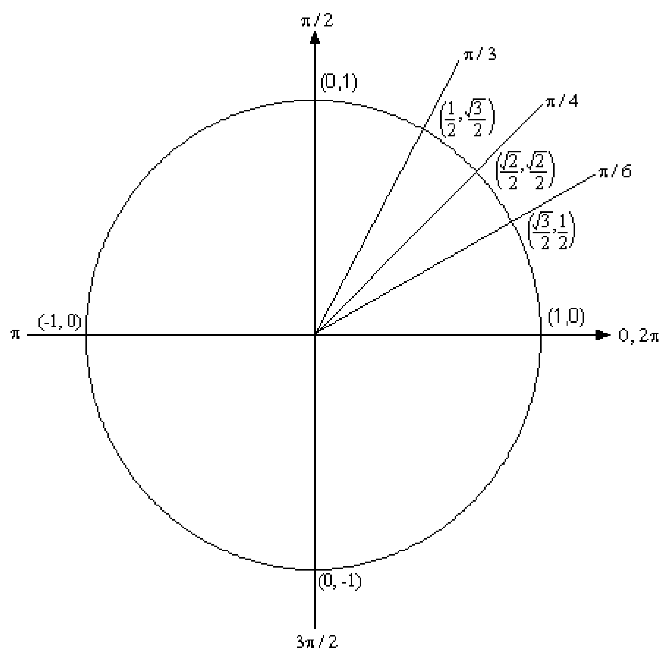
Degree	0	30	45	60	90	180	270	360
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

Know this table! There are, of course, many other angles in radians that we'll see during this class, but most will relate back to these few angles. So, if you can deal with these angles you will be able to deal with most of the others.

Be forewarned, everything in most calculus classes will be done in radians!

Let's now take a look at one of the most overlooked ideas from a trig class. The unit circle is one of the more useful tools to come out of a trig class. Unfortunately, most people don't learn it as well as they should in their trig class.

Below is the unit circle with just the first quadrant filled in. The way the unit circle works is to draw a line from the center of the circle outwards corresponding to a given angle. Then look at the coordinates of the point where the line and the circle intersect. The first coordinate is the cosine of that angle and the second coordinate is the sine of that angle. There are a couple of *basic* angles that are commonly used. These are $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \pi, \frac{3\pi}{2},$ and 2π and are shown below along with the coordinates of the intersections. So, from the unit circle below we can see that $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ and $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$.



Remember how the signs of angles work. If you rotate in a counter clockwise direction the angle is positive and if you rotate in a clockwise direction the angle is negative.

Recall as well that one complete revolution is 2π, so the positive x-axis can correspond to either an angle of 0 or 2π (or 4π, or 6π, or -2π, or -4π, *etc*. depending on the direction of rotation). Likewise, the angle $\frac{\pi}{6}$ (to pick an angle completely at random) can also be any of the following angles:

$$\frac{\pi}{6}+2\pi=\frac{13\pi}{6} \text{ (start at } \frac{\pi}{6} \text{ then rotate once around counter clockwise)}$$

$$\frac{\pi}{6}+4\pi=\frac{25\pi}{6} \text{ (start at } \frac{\pi}{6} \text{ then rotate around twice counter clockwise)}$$

$$\frac{\pi}{6}-2\pi=-\frac{11\pi}{6} \text{ (start at } \frac{\pi}{6} \text{ then rotate once around clockwise)}$$

$$\frac{\pi}{6}-4\pi=-\frac{23\pi}{6} \text{ (start at } \frac{\pi}{6} \text{ then rotate around twice clockwise)}$$

etc.

In fact $\frac{\pi}{6}$ can be any of the following angles $\frac{\pi}{6}+2\pi\,n$, $n=0,\pm 1,\pm 2,\pm 3,\ldots$ In this case n is the number of complete revolutions you make around the unit circle starting at $\frac{\pi}{6}$. Positive values of n correspond to counter clockwise rotations and negative values of n correspond to clockwise rotations.

So, why did I only put in the first quadrant? The answer is simple. If you know the first quadrant then you can get all the other quadrants from the first with a small application of geometry. You'll see how this is done in the following example.

Example 1 Evaluate each of the following.

(a) $\sin\left(\frac{2\pi}{3}\right)$ and $\sin\left(-\frac{2\pi}{3}\right)$.

(b) $\cos\left(\frac{7\pi}{6}\right)$ and $\cos\left(-\frac{7\pi}{6}\right)$

(c) $\tan\left(-\frac{\pi}{4}\right)$ and $\tan\left(\frac{7\pi}{4}\right)$

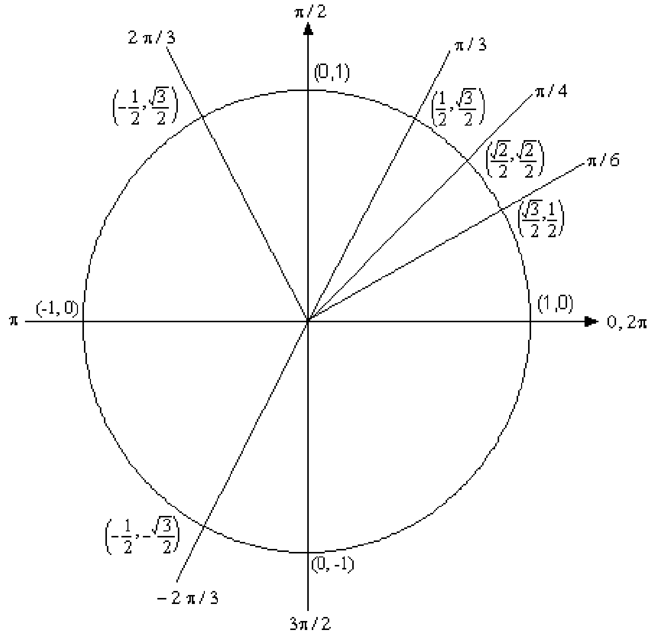
(d) $\sec\left(\frac{25\pi}{6}\right)$

Solution

(a) The first evaluation in this part uses the angle $\frac{2\pi}{3}$. That's not on our unit circle, however notice that $\frac{2\pi}{3}=\pi-\frac{\pi}{3}$. So $\frac{2\pi}{3}$ is found by rotating up $\frac{\pi}{3}$ from the negative *x*-axis. This means that the line for $\frac{2\pi}{3}$ will be a mirror image of the line for $\frac{\pi}{3}$ only in the second quadrant. The coordinates for $\frac{2\pi}{3}$ will be the coordinates for $\frac{\pi}{3}$ except the *x* coordinate will be negative.

Likewise for $-\frac{2\pi}{3}$ we can notice that $-\frac{2\pi}{3} = -\pi + \frac{\pi}{3}$, so this angle can be found by rotating down $\frac{\pi}{3}$ from the negative *x*-axis. This means that the line for $-\frac{2\pi}{3}$ will be a mirror image of the line for $\frac{\pi}{3}$ only in the third quadrant and the coordinates will be the same as the coordinates for $\frac{\pi}{3}$ except both will be negative.

Both of these angles along with their coordinates are shown on the following unit circle.



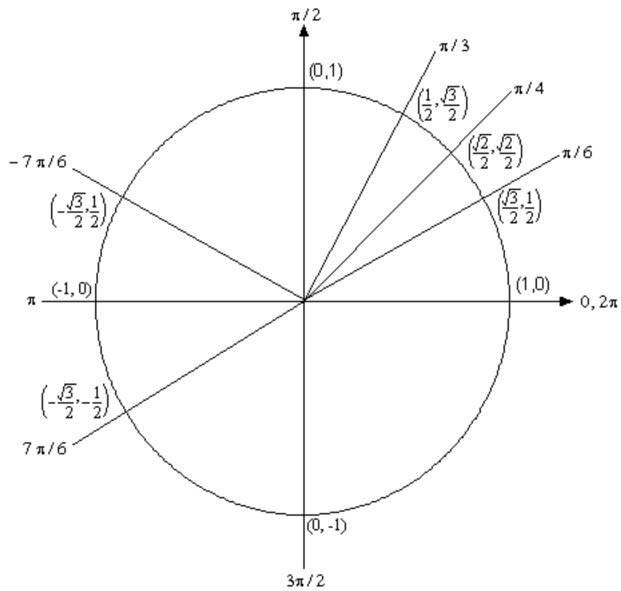
From this unit circle we can see that $\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$ and $\sin\left(-\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2}$.

This leads to a nice fact about the sine function. The sine function is called an **odd** function and so for ANY angle we have

$$\sin(-\theta) = -\sin(\theta)$$

(b) For this example notice that $\frac{7\pi}{6} = \pi + \frac{\pi}{6}$ so this means we would rotate down $\frac{\pi}{6}$ from the negative *x*-axis to get to this angle. Also $-\frac{7\pi}{6} = -\pi - \frac{\pi}{6}$ so this means we would rotate up $\frac{\pi}{6}$ from the negative *x*-axis to get to this angle. So, as with the last part, both of these angles will be mirror images of $\frac{\pi}{6}$ in the third and second quadrants respectively and we can use this to determine the coordinates for both of these new angles.

Both of these angles are shown on the following unit circle along with appropriate coordinates for the intersection points.

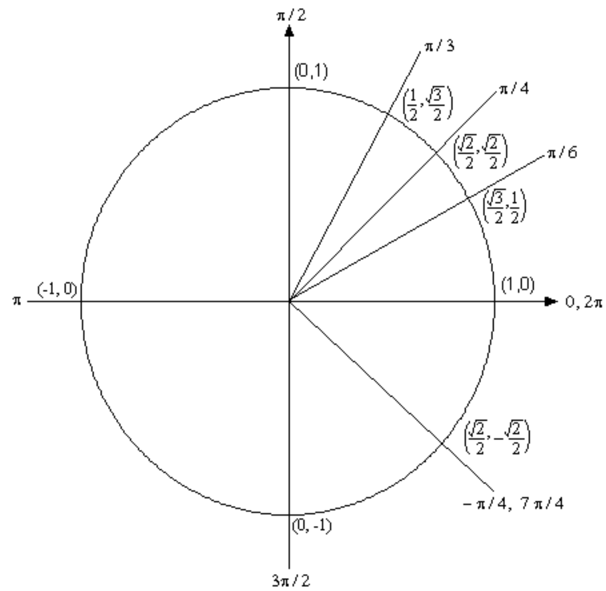


From this unit circle we can see that $\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$ and $\cos\left(-\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$. In this case the cosine function is called an **even** function and so for ANY angle we have

$$\cos(-\theta) = \cos(\theta).$$

(c) Here we should note that $\frac{7\pi}{4} = 2\pi - \frac{\pi}{4}$ so $\frac{7\pi}{4}$ and $-\frac{\pi}{4}$ are in fact the same angle!

Also note that this angle will be the mirror image of $\frac{\pi}{4}$ in the fourth quadrant. The unit circle for this angle is



Now, if we remember that $\tan\left(x\right)=\dfrac{\sin\left(x\right)}{\cos\left(x\right)}$ we can use the unit circle to find the values the tangent function. So,

$$\tan\left(\frac{7\pi}{6}\right)=\tan\left(-\frac{\pi}{4}\right)=\frac{\sin\left(-\pi/4\right)}{\cos\left(-\pi/4\right)}=\frac{-\sqrt{2}/2}{\sqrt{2}/2}=-1.$$

On a side note, notice that $\tan\left(\dfrac{\pi}{4}\right)=1$ and we can see that the tangent function is also called an **odd** function and so for ANY angle we will have

$$\tan\left(-\theta\right)=-\tan\left(\theta\right).$$

(d) Here we need to notice that $\dfrac{25\pi}{6}=4\pi+\dfrac{\pi}{6}$. In other words, we've started at $\dfrac{\pi}{6}$ and rotated around twice to end back up at the same point on the unit circle. This means that

$$\sec\left(\frac{25\pi}{6}\right)=\sec\left(4\pi+\frac{\pi}{6}\right)=\sec\left(\frac{\pi}{6}\right)$$

Now, let's also not get excited about the secant here. Just recall that

$$\sec\left(x\right)=\frac{1}{\cos\left(x\right)}$$

and so all we need to do here is evaluate a cosine! Therefore,

$$\sec\left(\frac{25\pi}{6}\right)=\sec\left(\frac{\pi}{6}\right)=\frac{1}{\cos\left(\frac{\pi}{6}\right)}=\frac{1}{\sqrt{3}/2}=\frac{2}{\sqrt{3}}$$

So, in the last example we saw how the unit circle can be used to determine the value of the trig functions at any of the "common" angles. It's important to notice that all of these examples used the fact that if you know the first quadrant of the unit circle and can relate all the other angles to "mirror images" of one of the first quadrant angles you don't really need to know whole unit circle. If you'd like to see a complete unit circle I've got one on my Trig Cheat Sheet that is available at http://tutorial.math.lamar.edu.

Another important idea from the last example is that when it comes to evaluating trig functions all that you really need to know is how to evaluate sine and cosine. The other four trig functions are defined in terms of these two so if you know how to evaluate sine and cosine you can also evaluate the remaining four trig functions.

We've not covered many of the topics from a trig class in this section, but we did cover some of the more important ones from a calculus standpoint. There are many important trig formulas that you will use occasionally in a calculus class. Most notably are the half-angle and double-angle formulas. If you need reminded of what these are, you might want to download my Trig Cheat Sheet as most of the important facts and formulas from a trig class are listed there.

Review : Solving Trig Equations

In this section we will take a look at solving trig equations. This is something that you will be asked to do on a fairly regular basis in my class.

Let's just jump into the examples and see how to solve trig equations.

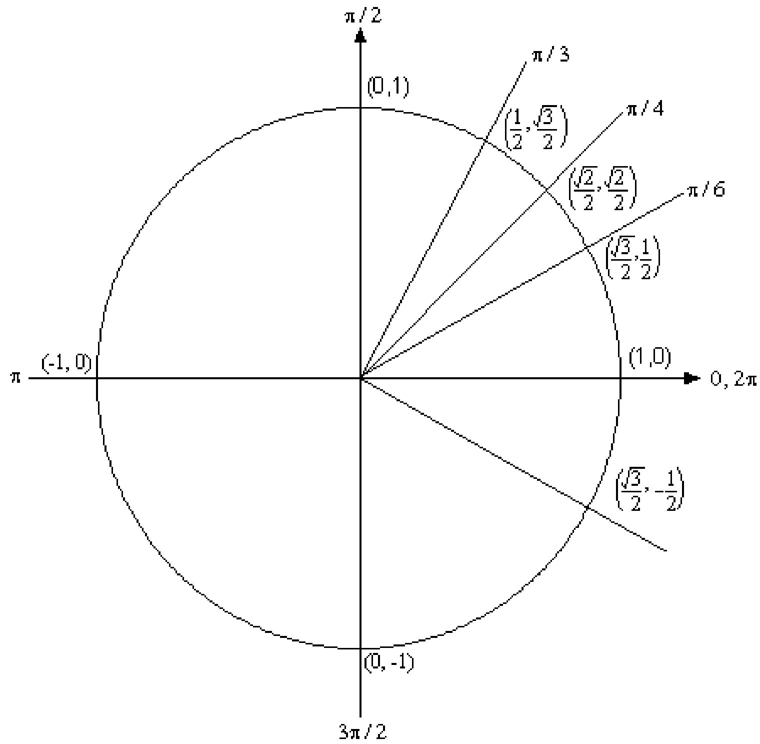
Example 1 Solve $2\cos\left(t\right)=\sqrt{3}$.

Solution

There's really not a whole lot to do in solving this kind of trig equation. All we need to do is divide both sides by 2 and the go to the unit circle.

$$\begin{aligned}2\cos\left(t\right)&=\sqrt{3}\\ \cos\left(t\right)&=\frac{\sqrt{3}}{2}\end{aligned}$$

So, we are looking for all the values of *t* for which cosine will have the value of $\frac{\sqrt{3}}{2}$. So, let's take a look at the following unit circle.



From quick inspection we can see that $t = \frac{\pi}{6}$ is a solution. However, as I have shown on the unit circle there is another angle which will also be a solution. We need to determine what this angle is. When we look for these angles we typically want *positive* angles that lie between 0 and 2π . This angle will not be the only possibility of course, but by convention we typically look for angles that meet these conditions.

To find this angle for this problem all we need to do is use a little geometry. The angle in the first quadrant makes an angle of $\frac{\pi}{6}$ with the positive *x*-axis, then so must the angle in the fourth quadrant. So we could use $-\frac{\pi}{6}$, but again, it's more common to use positive angles so, we'll use $t = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$.

We aren't done with this problem. As the discussion about finding the second angle has shown there are many ways to write any given angle on the unit circle. Sometimes it will be $-\frac{\pi}{6}$ that we want for the solution and sometimes we will want both (or neither) of the listed angles. Therefore, since there isn't anything in this problem (contrast this with the next problem) to tell us which is the correct solution we will need to list ALL possible solutions.

This is very easy to do. Recall from the previous section and you'll see there that I used

$$\frac{\pi}{6} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

to represent all the possible angles that can end at the same location on the unit circle, *i.e.* angles that end at $\frac{\pi}{6}$. Remember that all this says is that we start at $\frac{\pi}{6}$ then rotate around in the counter-clockwise direction (*n* is positive) or clockwise direction (*n* is negative) for *n* complete rotations. The same thing can be done for the second solution.

So, all together the complete solution to this problem is

$$\frac{\pi}{6} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$
$$\frac{11\pi}{6} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

As a final thought, notice that we can get $-\frac{\pi}{6}$ by using $n = -1$ in the second solution.

Now, in a calculus class this is not a typical trig equation that we'll be asked to solve. A more typical example is the next one.

Example 2 Solve $2\cos(t) = \sqrt{3}$ on $[-2\pi, 2\pi]$.

Solution
In a calculus class we are often more interested in only the solutions to a trig equation that fall in a certain interval.

The first step in this kind of problem is to first find all possible solutions. We did this in the first example.

$$\frac{\pi}{6} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$
$$\frac{11\pi}{6} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

Now, to find the solutions in the interval all we need to do is start picking values of *n*, plugging them in and getting the solutions that will fall into the interval that we've been given.

n=0.

$$\frac{\pi}{6} + 2\pi(0) = \frac{\pi}{6} < 2\pi$$
$$\frac{11\pi}{6} + 2\pi(0) = \frac{11\pi}{6} < 2\pi$$

Now, notice that if we take any positive value of *n* we will be adding on positive multiples of 2π onto a positive quantity and this will take us past the upper bound of our interval and so we don't need to take any positive value of *n*.

However, just because we aren't going to take any positive value of *n* doesn't mean that we shouldn't also look at negative values of *n*.

n=-1.

$$\frac{\pi}{6}+2\pi\left(-1\right)=-\frac{11\pi}{6}>-2\pi$$
$$\frac{11\pi}{6}+2\pi\left(-1\right)=-\frac{\pi}{6}>-2\pi$$

These are both greater than -2π and so are solutions, but if we subtract another 2π off (*i.e* use $n=-2$) we will once again be outside of the interval so we've found all the possible solutions that lie inside the interval $[-2\pi, 2\pi]$.

So, the solutions are : $\frac{\pi}{6}, \frac{11\pi}{6}, -\frac{\pi}{6}, -\frac{11\pi}{6}$.

So, let's see if you've got all this down.

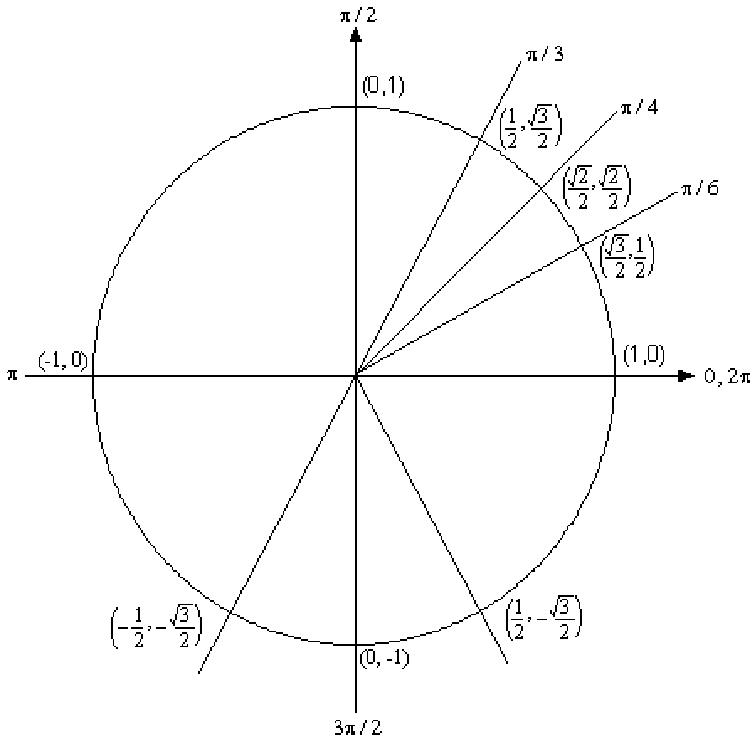
Example 3 Solve $2\sin\left(5x\right)=-\sqrt{3}$ on $[-\pi, 2\pi]$

Solution

This problem is very similar to the other problems in this section with a very important difference. We'll start this problem in exactly the same way. We first need to find all possible solutions.

$$2\sin(5x)=-\sqrt{3}$$
$$\sin(5x)=\frac{-\sqrt{3}}{2}$$

So, we are looking for angles that will give $-\frac{\sqrt{3}}{2}$ out of the sine function. Let's again go to our trusty unit circle.



Now, there are no angles in the first quadrant for which sine has a value of $-\frac{\sqrt{3}}{2}$. However, there are two angles in the lower half of the unit circle for which sine will have a value of $-\frac{\sqrt{3}}{2}$. So, what are these angles? We'll notice $\sin\left(\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2}$, so the angle in the third quadrant will be $\frac{\pi}{3}$ below the ***negative*** x-axis or $\pi+\frac{\pi}{3}=\frac{4\pi}{3}$. Likewise, the angle in the fourth quadrant will $\frac{\pi}{3}$ below the ***positive*** x-axis or $2\pi-\frac{\pi}{3}=\frac{5\pi}{3}$. Remember that we're typically looking for positive angles between 0 and 2π.

Now we come to the very important difference between this problem and the previous problems in this section. The solution is **NOT**

$$x=\frac{4\pi}{3}+2\pi n,\quad n=0,\pm1,\pm2,\ldots$$

$$x=\frac{5\pi}{3}+2\pi n,\quad n=0,\pm1,\pm2,\ldots$$

This is not the set of solutions because we are NOT looking for values of x for which $\sin(x)=-\frac{\sqrt{3}}{2}$, but instead we are looking for values of x for which $\sin(5x)=-\frac{\sqrt{3}}{2}$. Note the difference in the arguments of the sine function! One is x and the other is $5x$. This makes all the difference in the world in finding the solution! Therefore, the set of solutions is

$$5x = \frac{4\pi}{3} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$
$$5x = \frac{5\pi}{3} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$

Well, actually, that's not quite the solution. We are looking for values of x so divide everything by 5 to get.

$$x = \frac{4\pi}{15} + \frac{2\pi n}{5}, \quad n = 0, \pm 1, \pm 2, \ldots$$
$$x = \frac{\pi}{3} + \frac{2\pi n}{5}, \quad n = 0, \pm 1, \pm 2, \ldots$$

Notice that we also divided the $2\pi n$ by 5 as well! This is important! If we don't do that you **WILL** miss solutions. For instance, take $n = 1$.

$$x = \frac{4\pi}{15} + \frac{2\pi}{5} = \frac{10\pi}{15} = \frac{2\pi}{3} \qquad \Rightarrow \qquad \sin\left(5\left(\frac{2\pi}{3}\right)\right) = \sin\left(\frac{10\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$
$$x = \frac{\pi}{3} + \frac{2\pi}{5} = \frac{11\pi}{15} \qquad \Rightarrow \qquad \sin\left(5\left(\frac{11\pi}{15}\right)\right) = \sin\left(\frac{11\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

I'll leave it to you to verify my work showing they are solutions. However it makes the point. If you didn't divided the $2\pi n$ by 5 you would have missed these solutions!

Okay, now that we've gotten all possible solutions it's time to find the solutions on the given interval. We'll do this as we did in the previous problem. Pick values of n and get the solutions.

n=0.

$$x = \frac{4\pi}{15} + \frac{2\pi(0)}{5} = \frac{4\pi}{15} < 2\pi$$
$$x = \frac{\pi}{3} + \frac{2\pi(0)}{5} = \frac{\pi}{3} < 2\pi$$

n=1.

$$x = \frac{4\pi}{15} + \frac{2\pi(1)}{5} = \frac{2\pi}{3} < 2\pi$$
$$x = \frac{\pi}{3} + \frac{2\pi(1)}{5} = \frac{11\pi}{15} < 2\pi$$

n=2.

$$x = \frac{4\pi}{15} + \frac{2\pi(2)}{5} = \frac{16\pi}{15} < 2\pi$$
$$x = \frac{\pi}{3} + \frac{2\pi(2)}{5} = \frac{17\pi}{15} < 2\pi$$

n=3.

$$x = \frac{4\pi}{15} + \frac{2\pi(3)}{5} = \frac{22\pi}{15} < 2\pi$$

$$x = \frac{\pi}{3} + \frac{2\pi(3)}{5} = \frac{23\pi}{15} < 2\pi$$

n=4.

$$x = \frac{4\pi}{15} + \frac{2\pi(4)}{5} = \frac{28\pi}{15} < 2\pi$$

$$x = \frac{\pi}{3} + \frac{2\pi(4)}{5} = \frac{29\pi}{15} < 2\pi$$

n=5.

$$x = \frac{4\pi}{15} + \frac{2\pi(5)}{5} = \frac{34\pi}{15} > 2\pi$$

$$x = \frac{\pi}{3} + \frac{2\pi(5)}{5} = \frac{35\pi}{15} > 2\pi$$

Okay, so we finally got past the right endpoint of our interval so we don't need any more positive *n*. Now let's take a look at the negative *n* and see what we've got.

n=-1.

$$x = \frac{4\pi}{15} + \frac{2\pi(-1)}{5} = -\frac{2\pi}{15} > -\pi$$

$$x = \frac{\pi}{3} + \frac{2\pi(-1)}{5} = -\frac{\pi}{15} > -\pi$$

n=-2.

$$x = \frac{4\pi}{15} + \frac{2\pi(-2)}{5} = -\frac{8\pi}{15} > -\pi$$

$$x = \frac{\pi}{3} + \frac{2\pi(-2)}{5} = -\frac{7\pi}{15} > -\pi$$

n=-3.

$$x = \frac{4\pi}{15} + \frac{2\pi(-3)}{5} = -\frac{14\pi}{15} > -\pi$$

$$x = \frac{\pi}{3} + \frac{2\pi(-3)}{5} = -\frac{13\pi}{15} > -\pi$$

n=-4.

$$x = \frac{4\pi}{15} + \frac{2\pi(-4)}{5} = -\frac{4\pi}{5} < -\pi$$

$$x = \frac{\pi}{3} + \frac{2\pi(-4)}{5} = -\frac{19\pi}{15} < -\pi$$

And we're now past the left endpoint of the interval. Sometimes, there will be many solutions as there were in this example. Putting all of this together gives the following set of solutions that lie in the given interval.

$$\frac{4\pi}{15},\frac{\pi}{3},\frac{2\pi}{3},\frac{11\pi}{15},\frac{16\pi}{15},\frac{17\pi}{15},\frac{22\pi}{15},\frac{23\pi}{15},\frac{28\pi}{15},\frac{29\pi}{15}$$
$$-\frac{\pi}{15},-\frac{2\pi}{15},-\frac{7\pi}{15},-\frac{8\pi}{15},-\frac{13\pi}{15},-\frac{14\pi}{15}$$

Let's work another example.

Example 4 Solve $\sin\left(2x\right)=-\cos\left(2x\right)$ on $\left[-\frac{3\pi}{2},\frac{3\pi}{2}\right]$

Solution

This problem is a little different from the previous ones. First, we need to do some rearranging and simplification.

$$\sin(2x)=-\cos(2x)$$
$$\frac{\sin(2x)}{\cos(2x)}=-1$$
$$\tan\left(2x\right)=-1$$

So, solving $\sin(2x)=-\cos(2x)$ is the same as solving $\tan(2x)=-1$. At some level we didn't need to do this for this problem as all we're looking for is angles in which sine and cosine have the same value, but opposite signs. However, for other problems this won't be the case and we'll want to convert to tangent.

Looking at our trusty unit circle it appears that the solutions will be,

$$2x=\frac{3\pi}{4}+2\pi n,\quad n=0,\pm1,\pm2,\ldots$$
$$2x=\frac{7\pi}{4}+2\pi n,\quad n=0,\pm1,\pm2,\ldots$$

Or, upon dividing by the 2 we get all possible solutions.

$$x=\frac{3\pi}{8}+\pi n,\quad n=0,\pm1,\pm2,\ldots$$
$$x=\frac{7\pi}{8}+\pi n,\quad n=0,\pm1,\pm2,\ldots$$

Now, let's determine the solutions that lie in the given interval.

n=0.

$$x=\frac{3\pi}{8}+\pi\left(0\right)=\frac{3\pi}{8}<\frac{3\pi}{2}$$
$$x=\frac{7\pi}{8}+\pi\left(0\right)=\frac{7\pi}{8}<\frac{3\pi}{2}$$

n=1.

$$x = \frac{3\pi}{8} + \pi\left(1\right) = \frac{11\pi}{8} < \frac{3\pi}{2}$$
$$x = \frac{7\pi}{8} + \pi\left(1\right) = \frac{15\pi}{8} > \frac{3\pi}{2}$$

Unlike the previous example only one of these will be in the interval. This will happen occasionally so don't always expect both answers from a particular *n* to work. Also, we should now check *n=2* for the first to see if it will be in or out of the interval. I'll leave it to you to check that it's out of the interval.

Now, let's check the negative *n*.

n=-1.

$$x = \frac{3\pi}{8} + \pi\left(-1\right) = -\frac{5\pi}{8} > -\frac{3\pi}{2}$$
$$x = \frac{7\pi}{8} + \pi\left(-1\right) = -\frac{\pi}{8} > -\frac{3\pi}{2}$$

n=-2.

$$x = \frac{3\pi}{8} + \pi\left(-2\right) = -\frac{13\pi}{8} < -\frac{3\pi}{2}$$
$$x = \frac{7\pi}{8} + \pi\left(-2\right) = -\frac{9\pi}{8} > -\frac{3\pi}{2}$$

Again, only one will work here. I'll leave it to you to verify that *n=-3* will give two answers that are both out of the interval.

The complete list of solutions is then,

$$-\frac{9\pi}{8}, -\frac{5\pi}{8}, -\frac{\pi}{8}, \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}$$

Let's work one more example so that I can make a point.

Example 5 Solve $\cos\left(3x\right) = 2$.

Solution

This is an example that is designed to remind you of certain properties about sine and cosine. Recall that $-1 \le \cos\left(\theta\right) \le 1$ and $-1 \le \sin\left(\theta\right) \le 1$. Therefore, since cosine will never be greater that 1 it definitely can't be 2. So **THERE ARE NO SOLUTIONS** to this equation!

In this section we solved some simple trig equations. There are more complicated trig equations that we can solve so don't leave this section with the feeling that there is nothing harder out there in the world to solve. If you would like to see a couple of more complicated problems you should check out my Algebra Trig Review at http://tutorial.math.lamar.edu. I've got a couple of additional problems there.

Review : Exponential Functions

In this section we're going to review one of the more common functions in both calculus and the sciences. However, before getting to this function let's take a much more general approach to things.

Let's start with $b>0$, $b \neq 1$. An exponential function is then a function in the form,

$$f(x) = b^x$$

Note that we avoid $b=1$ because that would give the constant function, $f(x)=1$. We avoid $b=0$ since this would also give a constant function and we avoid negative values of b for the following reason. Let's, for a second, suppose that we did allow b to be negative and look at the following function.

$$g(x) = (-4)^x$$

Let's do some evaluation.

$$g(2) = (-4)^2 = 16 \qquad\qquad g\left(\frac{1}{2}\right) = -(-4)^{\frac{1}{2}} = \sqrt{-4} = 2i$$

So, for some values of x we will get real numbers and for other values of x we well get complex numbers. We want to avoid this and so if we require $b>0$ this will not be a problem.

Let's take a look at a couple of exponential functions.

Example 1 Sketch the graph of $f(x) = 2^x$ and $g(x) = \left(\frac{1}{2}\right)^x$

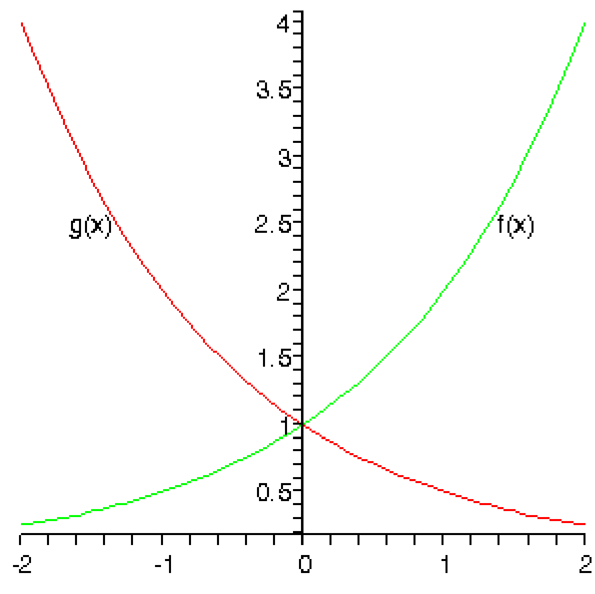
Solution

Let's first get a table of values for these two functions.

x	$f(x)$	$g(x)$
-2	$f(-2) = 2^{-2} = \frac{1}{4}$	$g(-2) = \left(\frac{1}{2}\right)^{-2} = 4$
-1	$f(-1) = 2^{-1} = \frac{1}{2}$	$g(-1) = \left(\frac{1}{2}\right)^{-1} = 2$
0	$f(0) = 2^0 = 1$	$g(0) = \left(\frac{1}{2}\right)^0 = 1$
1	$f(1) = 2$	$g(1) = \frac{1}{2}$

2	$f(2)=4$	$g(2)=\frac{1}{4}$

Here's the sketch of both of these functions.



This graph illustrates some very nice properties about exponential functions in general.

Properties of $f(x)=b^x$

1. $f(0)=1$. The function will always take the value of 1 at *x=0*.
2. $f(x)\neq 0$. An exponential function will never be zero.
3. $f(x)>0$. An exponential function is always positive.
4. The previous two properties can be summarized by saying that the range of an exponential function is $(0,\infty)$.
5. The domain of an exponential function is $(-\infty,\infty)$. In other words, you can plug every *x* into an exponential function.
6. If $0<b<1$ then,
   a. $f(x)\rightarrow 0$ as $x\rightarrow\infty$
   b. $f(x)\rightarrow\infty$ as $x\rightarrow-\infty$
7. If $b>1$ then,
   a. $f(x)\rightarrow\infty$ as $x\rightarrow\infty$
   b. $f(x)\rightarrow 0$ as $x\rightarrow-\infty$

These will all be very useful properties to recall at times as we move throughout this course (and later Calculus courses for that matter…).

There is a very important exponential function that arises naturally in many places. This function is called the **natural exponential function**. However, for must people this is simply the exponential function.

Definition : The **natural exponential function** is $f(x) = \mathbf{e}^x$ where, $\mathbf{e} = 2.71828182845905\ldots$.

So, since **e**>*1* we also know that $\mathbf{e}^x \to \infty$ as $x \to \infty$ and $\mathbf{e}^x \to 0$ as $x \to -\infty$.

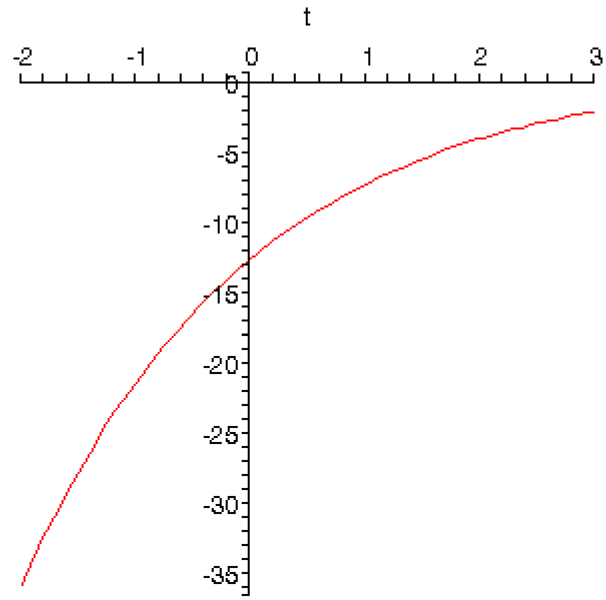
Let's take a quick look at an example.

Example 2 Sketch the graph of $h(t) = 1 - 5\mathbf{e}^{1-\frac{t}{2}}$

Solution
Let's first get a table of values for this function.

x	-2	-1	0	1	2	3
f(x)	-35.9453	-21.4084	-12.5914	-7.2436	-4	-2.0327

Here is the sketch.



The main point behind this problem is to make sure you can do this type of evaluation so make sure that you can get the values that we graphed in this example. You will be asked to do this kind of evaluation on occasion in this class.

You will be seeing exponential functions in every chapter in this class so make sure that you are comfortable with them.

Review : Logarithm Functions

In this section we'll take a look at a function that is related to the exponential functions we looked at in the last section. We will look logarithms in this section. Logarithms are one of the functions that students fear the most. The main reason for this seems to be that they simply have never really had to work with them. Once they start working with them, students come to realize that they aren't as bad as they first thought.

We'll start with $b>0$, $b \neq 1$ just as we did in the last section. Then we have

$$y = \log_b x \qquad \text{is equivalent to} \qquad x = b^y$$

The first is called logarithmic form and the second is called the exponential form. Remembering this equivalence is the key to evaluating logarithms. The number, *b*, is called the base.

Example 1 Without a calculator give the exact value of each of the following logarithms.

(a) $\log_2 16$

(b) $\log_4 16$

(c) $\log_5 625$

(d) $\log_9 \dfrac{1}{531441}$

(e) $\log_{\frac{1}{6}} 36$

(f) $\log_{\frac{3}{2}} \dfrac{27}{8}$

Solution

To quickly evaluate logarithms the easiest thing to do is to convert the logarithm to exponential form. So, let's take a look at the first one.

(a) First, let's "convert" to exponential form.

$$\log_2 16 = ? \qquad \text{is equivalent to} \qquad 2^? = 16$$

So, we're really asking 2 raised to what gives 16. Since 2 raised to 4 is 16 we get,

$$\log_2 16 = 4 \qquad \text{because} \qquad 2^4 = 16$$

We'll not do the remainders in quite this detail, but they were all worked in this way.

(b) $\log_4 16 = 2$ because $4^2 = 16$

Note the difference the first and second logarithm! The base is important! It can completely change the answer.

(c) $\log_5 625 = 4$ because $5^4 = 625$

(d) $\log_9 \dfrac{1}{531441} = -6$ because $9^{-6} = \dfrac{1}{9^6} = \dfrac{1}{531441}$

(e) $\log_{\frac{1}{6}} 36 = -2$ because $\left(\dfrac{1}{6}\right)^{-2} = 6^2 = 36$

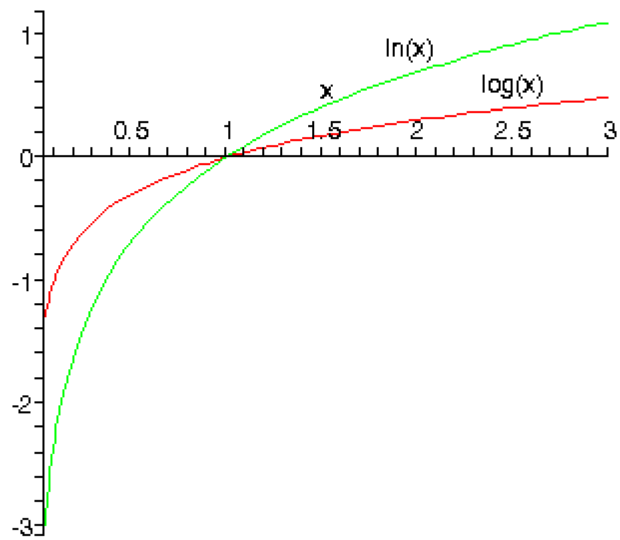
(f) $\log_{\frac{3}{2}} \dfrac{27}{8} = 3$ because $\left(\dfrac{3}{2}\right)^3 = \dfrac{27}{8}$

There are a couple of special logarithms that arise in many places. These are,

$\ln x = \log_{\mathbf{e}} x$ This log is called the natural logarithm

$\log x = \log_{10} x$ This log is called the common logarithm

In the natural logarithm the base **e** is the same number as in the natural exponential logarithm that we saw in the last section. Here is a sketch of both of these logarithms.



From this graph we can get a couple of very nice properties about the natural logarithm that we will use many times in this and later Calculus courses.

$$\ln x \to \infty \quad \text{as} \quad x \to \infty$$

$$\ln x \to -\infty \quad \text{as} \quad x \to 0,\ x > 0$$

Let's take a look at a couple of more logarithm evaluations. Some of which deal with the natural or common logarithm and some of which don't.

Example 2 Without a calculator give the exact value of each of the following logarithms.

(a) $\ln \sqrt[3]{\mathbf{e}}$

(b) $\log 1000$

(c) $\log_{16} 16$

(d) $\log_{23} 1$

(e) $\log_2 \sqrt[7]{32}$

Solution

These work exactly the same as previous example so we won't put in too many details.

(a) $\ln \sqrt[3]{\mathbf{e}} = \frac{1}{3}$ because $\mathbf{e}^{\frac{1}{3}} = \sqrt[3]{\mathbf{e}}$

(b) $\log 1000 = 3$ because $10^3 = 1000$

(c) $\log_{16} 16 = 1$ because $16^1 = 16$

(d) $\log_{23} 1 = 0$ because $23^0 = 1$

(e) $\log_2 \sqrt[7]{32} = \frac{5}{7}$ because $\sqrt[7]{32} = 32^{\frac{1}{7}} = \left(2^5\right)^{\frac{1}{7}} = 2^{\frac{5}{7}}$

This last set of examples leads us to some of the basic properties of logarithms.

Properties

1. The domain of the logarithm function is $(0, \infty)$. In other words, we can only plug positive numbers into a logarithm! We can't plug in zero or a negative number.
2. $\log_b b = 1$
3. $\log_b 1 = 0$
4. $\log_b b^x = x$
5. $b^{\log_b x} = x$

The last two properties will be especially useful in the next section. Notice as well that these last two properties tell us that,

$$f(x) = b^x \qquad \text{and} \qquad g(x) = \log_b x$$

are inverses of each other.

Here are some more properties that are useful in the manipulation of logarithms.

More Properties

6. $\log_b xy = \log_b x + \log_b y$
7. $\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$
8. $\log_b \left(x^r\right) = r \log_b x$

Note that there is no equivalent property to the first two for sums and differences.

$$\log_b(x+y) \neq \log_b x + \log_b y$$
$$\log_b(x-y) \neq \log_b x - \log_b y$$

Example 3 Write each of the following in terms of simpler logarithms.

(a) $\ln x^3 y^4 z^5$

(b) $\log_3\left(\dfrac{9x^4}{\sqrt{y}}\right)$

(c) $\log\left(\dfrac{x^2+y^2}{(x-y)^3}\right)$

Solution

What the instructions really mean here is to use as many if the properties of logarithms as we can to simplify things down as much as we can.

(a) Property 6 above can be extended to products of more than two functions. Once we've used Property 6 we can then use Property 8.

$$\begin{aligned}\ln x^3 y^4 z^5 &= \ln x^3 + \ln y^4 + \ln z^5 \\ &= 3\ln x + 4\ln y + 5\ln z\end{aligned}$$

(b) When using property 7 above make sure that the logarithm that you subtract is the one that contains the denominator as its argument. Also, note that that we'll be converting the root to fractional exponents in the first step.

$$\begin{aligned}\log_3\left(\frac{9x^4}{\sqrt{y}}\right) &= \log_3 9x^4 - \log_3 y^{\frac{1}{2}} \\ &= \log_3 9 + \log_3 x^4 - \log_3 y^{\frac{1}{2}} \\ &= 2 + 4\log_3 x - \frac{1}{2}\log_3 y\end{aligned}$$

(c) The point to this problem is mostly the correct use of property 8 above.

$$\begin{aligned}\log\left(\frac{x^2+y^2}{(x-y)^3}\right) &= \log\left(x^2+y^2\right) - \log(x-y)^3 \\ &= \log\left(x^2+y^2\right) - 3\log(x-y)\end{aligned}$$

You can use Property 8 on the second term because the WHOLE term was raised to the 3, but in the first logarithm, only the individual terms were squared and not the term as a whole so the 2's must stay where they are!

The last topic that we need to look at in this section is the **change of base** formula for logarithms. The change of base formula is,

$$\log_b x = \frac{\log_a x}{\log_a b}$$

This is the most general change of base formula and will convert from base *b* to base *a*. However, the usual reason for using the change of base formula is to compute the value of a logarithm that is in a base that you can't easily deal with. Using the change of base formula means that you can write the logarithm in terms of a logarithm that you can deal with. The two most common change of base formulas are

$$\log_b x = \frac{\ln x}{\ln b} \qquad \text{and} \qquad \log_b x = \frac{\log x}{\log b}$$

In fact, often you will see one or the other listed as THE change of base formula!

In the first part of this section we computed the value of a few logarithms, but we could do these without the change of base formula because all the arguments could be written in terms of the base to a power. For instance,

$$\log_7 49 = 2 \qquad \text{because} \qquad 7^2 = 49$$

However, this only works because 49 can be written as a power of 7! We would need the change of base formula to compute $\log_7 50$.

$$\log_7 50 = \frac{\ln 50}{\ln 7} = \frac{3.91202300543}{1.94591014906} = 2.0103821378$$

OR

$$\log_7 50 = \frac{\log 50}{\log 7} = \frac{1.69897000434}{0.845098040014} = 2.0103821378$$

So, it doesn't matter which we use, we will get the same answer regardless of the logarithm that we use in the change of base formula.

Note as well that we could use the change of base formula on $\log_7 49$ if we wanted to as well.

$$\log_7 49 = \frac{\ln 49}{\ln 7} = \frac{3.89182029811}{1.94591014906} = 2$$

This is a lot of work however, and is probably not the best way to deal with this.

So, in this section we saw how logarithms work and took a look at some of the properties of logarithms. We will run into logarithms on occasion so make sure that you can deal with them when we do run into them.

Review : Exponential and Logarithm Equations

In this section we'll take a look at solving equations with exponential functions or logarithms in them.

We'll start with equations that involve exponential functions. The main property that we'll need for these equations is,

$$\log_b b^x = x$$

Example 1 Solve $7+15\mathbf{e}^{1-3z}=10$.

Solution
The first step is to get the exponential all by itself on one side of the equation with a coefficient of one.

$$\begin{aligned} 7+15\mathbf{e}^{1-3z} &= 10 \\ 15\mathbf{e}^{1-3z} &= 3 \\ \mathbf{e}^{1-3z} &= \frac{1}{5} \end{aligned}$$

Now, we need to get the z out of the exponent so we can solve for it. To do this we will use the property above. Since we have an **e** in the equation we'll use the natural logarithm. First we take the logarithm of both sides and then use the property to simplify the equation.

$$\begin{aligned} \ln\left(\mathbf{e}^{1-3z}\right) &= \ln\left(\frac{1}{5}\right) \\ 1-3z &= \ln\left(\frac{1}{5}\right) \end{aligned}$$

All we need to now is solve this equation for z.

$$\begin{aligned} 1-3z &= \ln\left(\frac{1}{5}\right) \\ -3z &= -1+\ln\left(\frac{1}{5}\right) \\ z &= -\frac{1}{3}\left(-1+\ln\left(\frac{1}{5}\right)\right) \\ z &= 0.8698126372 \end{aligned}$$

Example 2 Solve $10^{t^2-t}=100$.

Solution
Now, in this case it looks like the best logarithm to use is the common logarithm since left hand side has a base of 10. There's no initial simplification to do, so just take the log of both sides and simplify.

$$\begin{aligned} \log 10^{t^2-t} &= \log 100 \\ t^2-t &= 2 \end{aligned}$$

At this point, we've just got a quadratic that can be solved

$$t^2 - t - 2 = 0$$
$$(t-2)(t+1) = 0$$
So, it looks like the solutions in this case are $t = 2$ and $t = -1$.

Now that we've seen a couple of equations where the variable only appears in the exponent we need to see an example with variables both in the exponent and out of it.

Example 3 Solve $x - x\mathbf{e}^{5x+2} = 0$.

Solution
The first step is to factor an *x* out of both terms.

DO NOT DIVIDE AN *x* FROM BOTH TERMS!!!!

$$x - x\mathbf{e}^{5x+2} = 0$$
$$x\left(1 - \mathbf{e}^{5x+2}\right) = 0$$
So, it's now a little easier to deal with. From this we can see that we get one of two possibilities.
$$x = 0 \qquad \text{OR}$$
$$1 - \mathbf{e}^{5x+2} = 0$$
The first possibility has nothing more to do, except notice that if we had divided both sides by an *x* we would have missed this one so be careful. In the second possibility we've got a little more to do. This is an equation similar to the first two that we did in this section.
$$\mathbf{e}^{5x+2} = 1$$
$$5x + 2 = \ln 1$$
$$5x + 2 = 0$$
$$x = -\frac{2}{5}$$
Don't forget that $\ln 1 = 0$!

So, the two solutions are $x = 0$ and $x = -\frac{2}{5}$.

Now let's take a look at some equations that involve logarithms. The main property that we'll be using to solve these kinds of equations is,
$$b^{\log_b x} = x$$

Example 4 Solve $3 + 2\ln\left(\frac{x}{7} + 3\right) = -4$.

Solution

This first step in this problem is to get the logarithm by itself on one side of the equation with a coefficient of 1.

$$2\ln\left(\frac{x}{7}+3\right)=-7$$
$$\ln\left(\frac{x}{7}+3\right)=-\frac{7}{2}$$

So using the property above with **e**, since there is a natural logarithm in the equation, we get,

$$\mathbf{e}^{\ln\left(\frac{x}{7}+3\right)}=\mathbf{e}^{-\frac{7}{2}}$$
$$\frac{x}{7}+3=\mathbf{e}^{-\frac{7}{2}}$$

Now all that we need to do is solve this for *x*.

$$\frac{x}{7}+3=\mathbf{e}^{-\frac{7}{2}}$$
$$\frac{x}{7}=-3+\mathbf{e}^{-\frac{7}{2}}$$
$$x=7\left(-3+\mathbf{e}^{-\frac{7}{2}}\right)$$
$$x=-20.78861832$$

At this point we might be tempted to say that we're done and move on. However, we do need to be careful. Recall from the previous section that we can't plug a negative number into a logarithm. This, by itself, doesn't mean that our answer won't work since its negative. What we need to do is plug it into the logarithm and make sure that $\frac{x}{7}+3$ will not be negative. I'll leave it to you to verify that this is in fact positive upon plugging our solution into the logarithm.

Let's now take a look at a more complicated equation. Often there will be more than one logarithm in the equation. When this happens we will need to use on or more of the following to combine all the logarithms into a single logarithm. Once this has been done we can proceed as we did in the previous example.

$$\log_b xy=\log_b x+\log_b y \qquad \log_b\left(\frac{x}{y}\right)=\log_b x-\log_b y \qquad \log_b\left(x^r\right)=r\log_b x$$

Example 5 Solve $2\ln\left(\sqrt{x}\right)-\ln\left(1-x\right)=2$.

Solution

First get the two logarithms combined into a single logarithm.

$$2\ln\left(\sqrt{x}\right)-\ln\left(1-x\right)=2$$
$$\ln\left(\left(\sqrt{x}\right)^2\right)-\ln\left(1-x\right)=2$$
$$\ln\left(x\right)-\ln\left(1-x\right)=2$$
$$\ln\left(\frac{x}{1-x}\right)=2$$

Now, exponentiate both sides and solve for *x*.

$$\frac{x}{1-x}=\mathbf{e}^2$$
$$x=\mathbf{e}^2\left(1-x\right)$$
$$x=\mathbf{e}^2-\mathbf{e}^2x$$
$$x\left(1+\mathbf{e}^2\right)=\mathbf{e}^2$$
$$x=\frac{\mathbf{e}^2}{1+\mathbf{e}^2}$$
$$x=0.8807970780$$

Finally, we just need to make sure that the solution, $x=0.8807970780$, doesn't produce negative numbers in both of the original logarithms. It doesn't, so this is in fact our solution to this problem.

Let's take a look at one more example.

Example 6 Solve $\log x+\log\left(x-3\right)=1$.

Solution

As with the last example, first combine the logarithms into a single logarithm.

$$\log x+\log\left(x-3\right)=1$$
$$\log\left(x\left(x-3\right)\right)=1$$

Now exponentiate both sides.

$$10^{\log\left(x^2-3x\right)}=10^1$$
$$x^2-3x=10$$
$$x^2-3x-10=0$$
$$\left(x-5\right)\left(x+2\right)=0$$

So, potential solutions are $x=5$ and $x=-2$. Note, however that if we plug $x=-2$ into either of the two original logarithms we would get negative numbers so this can't be a solution. We can however, use $x = 5$.

Therefore, the solution to this equation is $x = 5$.

When solving equations with logarithms it is important to check your potential solutions to make sure that they don't generate logarithms of negative numbers or zero. It is also important to make sure that you do the checks in the **original** equation. If you check them in the second logarithm above (after we've combined the two logs) both solutions will appear to work! This is because in combining the two logarithms we've actually changed the problem. In fact, it is this change that introduces the extra solution that we couldn't use!

Also be careful in solving equations containing logarithms to not get locked into the idea that you will get two potential solutions and only one of these will work. It is possible to have problems where both are solutions and where neither are solutions.

Review : Common Graphs

The purpose of this section is to make sure that you're familiar with the graphs of many of the basic functions that you're liable to run across in a calculus class.

Example 1 Graph $y = -\frac{2}{5}x + 3$.

Solution

This is a line in the slope intercept form

$$y = mx + b$$

In this case the line has a *y* intercept of (0,*b*) and a slope of *m*. Recall that slope can be thought of as

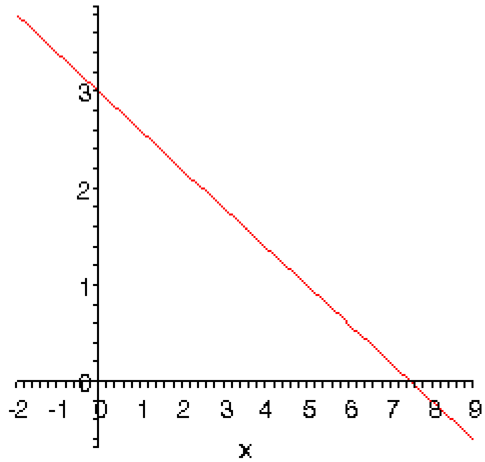
$$m = \frac{\text{rise}}{\text{run}}$$

If the slope is negative we tend to think of the rise as a fall.

The slope allows us to get a second point on the line. Once we have any point on the line and the slope we move right by *run* and up/down by *rise* depending on the sign. This will be a second point on the line.

In this case we know (0,3) is a point on the line and the slope is $-\frac{2}{5}$. So starting at (0,3) we'll move 5 to the right (*i.e.* $0 \to 5$) and down 2 (*i.e.* $3 \to 1$) to get (5,1) as a second point on the line. Once we've got two points on a line all we need to do is plot the two points and connect them with a line.

Here's the sketch for this line.



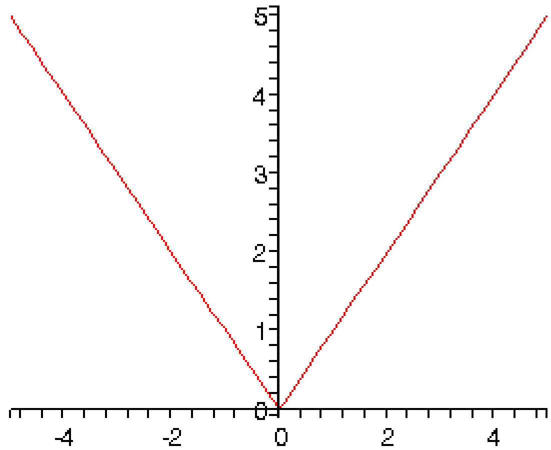
Example 2 Graph $f(x) = |x|$

Solution

There really isn't much to this problem outside of reminding ourselves of what absolute value is. Recall that the absolute value function is defined as,

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

The graph is then,



Example 3 Graph $f(x) = -x^2 + 2x + 3$.

Solution

This is a parabola in the general form.

$$f(x) = ax^2 + bx + c$$

In this form, the x-coordinate of the vertex (the highest or lowest point on the parabola) is

$x = -\frac{b}{2a}$ and we get the *y*-coordinate is $y = f\left(-\frac{b}{2a}\right)$. So, for our parabola the coordinates of the vertex will be.

$$x = -\frac{2}{2\left(-1\right)} = 1$$

$$y = f\left(1\right) = -\left(1\right)^2 + 2\left(1\right) + 3 = 4$$

So, the vertex for this parabola is (1,4).

We can also determine which direction the parabola opens from the sign of *a*. If *a* is positive the parabola opens up and if *a* is negative the parabola opens down. In our case the parabola opens down.

This also means that we'll have *x*-intercepts on this graph. So, we'll solve the following.

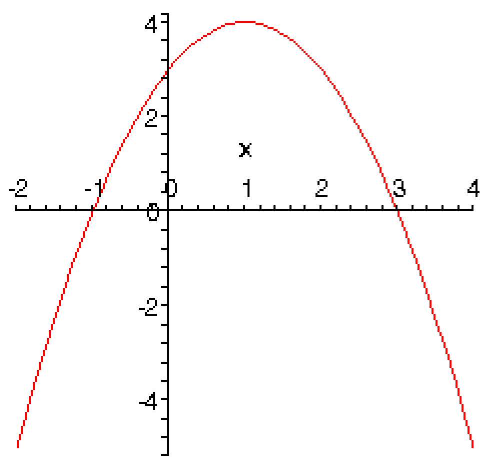
$$-x^2 + 2x + 3 = 0$$

$$x^2 - 2x - 3 = 0$$

$$\left(x-3\right)\left(x+1\right) = 0$$

So, we will have *x*-intercepts at $x = -1$ and $x = 3$. Notice that to make our life easier in the solution process we multiplied everything by -1 to get the coefficient of the x^2 positive. This made the factoring easier.

Here's a sketch of this parabola.



Example 4 Graph $f\left(y\right) = y^2 - 6y + 5$

Solution

Most people come out of an Algebra class capable of dealing with functions in the form $y = f(x)$. However, many functions that you will have to deal with in a Calculus class are in the form $x = f(y)$ and can only be easily worked with in that form. So, you need to get used to working with functions in this form.

The nice thing about these kinds of function is that if you can deal with functions in the form $y = f(x)$ then you can deal with functions in the form $x = f(y)$ even if you aren't that familiar with them.

Let's first consider the equation.

$$y = x^2 - 6x + 5$$

This is a parabola that opens up and has a vertex of (3,-4).

Well our function is in the form

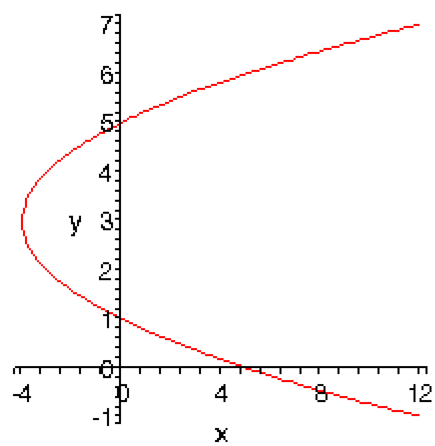
$$x = ay^2 + by + c$$

and this is also a parabola that opens left or right depending on the sign of *a*. The *y*-coordinate of the vertex is given by $y = -\frac{b}{2a}$ and we find the *x*-coordinate by plugging this into the equation.

Our function is a parabola that opens to the right (*a* is positive) and has a vertex at (-4,3). To graph this we'll need *y*-intercepts. We find these just like we found *x*-intercepts in the previous couple of problems.

$$y^2 - 6y + 5 = 0$$
$$(y-5)(y-1) = 0$$

The parabola will have *y*-intercepts at $y = 1$ and $y = 5$. Here's a sketch of the graph.



Example 5 Graph $x^2 + 2x + y^2 - 8y + 8 = 0$.

Solution

To determine just what kind of graph we've got here we need complete the square on both the *x* and the *y*.

$$\begin{aligned} x^2+2x+y^2-8y+8&=0 \\ x^2+2x+1-1+y^2-8y+16-16+8&=0 \\ (x+1)^2+(y-4)^2&=9 \end{aligned}$$

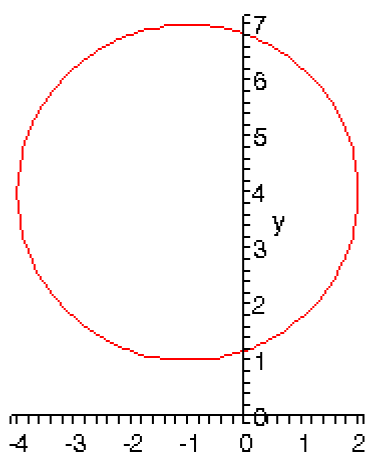
Recall that to complete the square we take the half of the coefficient of the *x* (or the *y*), square this and then add and subtract it to the equation.

Upon doing this we see that we have a circle and it's now written in standard form.

$$(x-h)^2+(y-k)^2=r^2$$

When circles are in this form we can easily identify the center : (*h*, *k*) and radius : *r*. Once we have these we can graph the circle simply by starting at the center and moving right, left, up and down by *r* to get the rightmost, leftmost, top most and bottom most points respectively.

Our circle has a center at (-1, 4) and a radius of 3. Here's a sketch of this circle.



Example 6 Graph $\dfrac{(x-2)^2}{9}+4(y+2)^2=1$

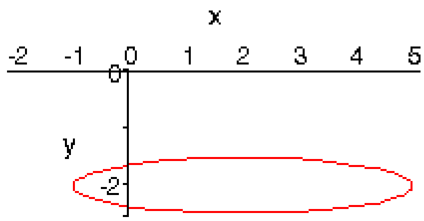
Solution

This is an ellipse. The standard form of the ellipse is

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

This is an ellipse with center (*h*, *k*) and the right most and left most points are a distance of *a* away from the center and the top most and bottom most points are a distance of *b* away from the center.

The ellipse for this problem has center (2, -2) and has $a=3$ and $b=\frac{1}{2}$. Here's a sketch of the ellipse.



Example 7 Graph $\frac{(x+1)^2}{9} - \frac{(y-2)^2}{4} = 1$

Solution

This is a hyperbola. There are actually two standard forms for a hyperbola.

Form	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$
Center	(h, k)	(h, k)
Opens	Opens right and left	Opens up and down
Vertices	*a* units right and left of center.	*b* units up and down from center.
Slope of Asymptotes	$\pm\frac{b}{a}$	$\pm\frac{b}{a}$

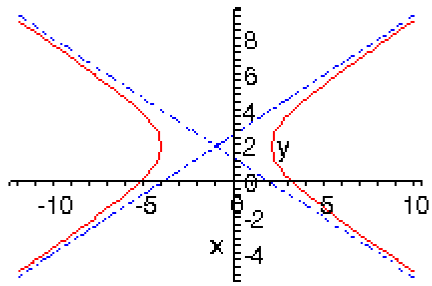
So, what does all this mean? First, notice that one of the terms is positive and the other is negative. This will determine which direction the two parts of the hyperbola open. If the x term is positive the hyperbola opens left and right. Likewise, if the y term is positive the parabola opens up and down.

Both have the same "center". Note that hyperbolas don't really have a center in the sense that circles and ellipses have centers. The center is the starting point in graphing a hyperbola. It tells up how to get to the vertices and how to get the asymptotes set up.

The asymptotes of a hyperbola are two lines that intersect at the center and have the slopes listed above. As you move farther out from the center the graph will get closer and closer to the asymptotes.

For the equation listed here the hyperbola will open left and right. Its center is (-1, 2). The two vertices are (-4, 2) and (2, 2). The asymptotes will have slopes $\pm\frac{2}{3}$.

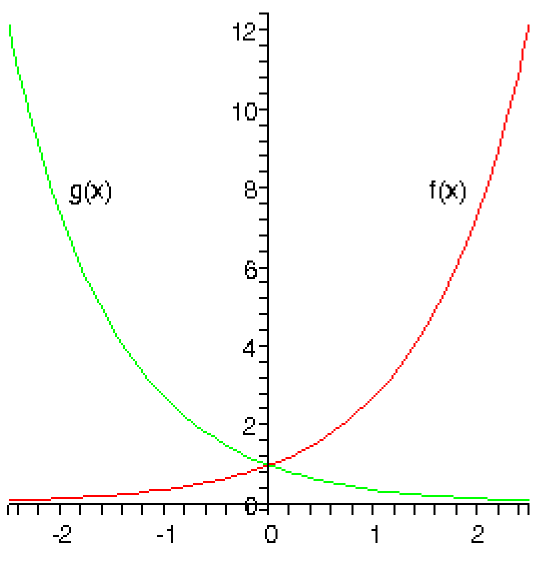
Here is a sketch of this hyperbola.



Example 8 Graph $f(x) = \mathbf{e}^{x}$ and $g(x) = \mathbf{e}^{-x}$

Solution

There really isn't a lot to this problem other than making sure these are graphed somewhere so that we can always remember how they behave.

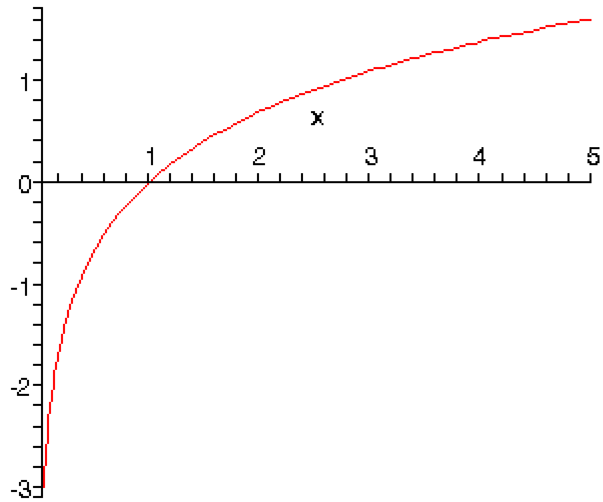


The behavior of these two exponential functions will be important on occasion.

Example 9 Graph $f(x) = \ln(x)$.

Solution

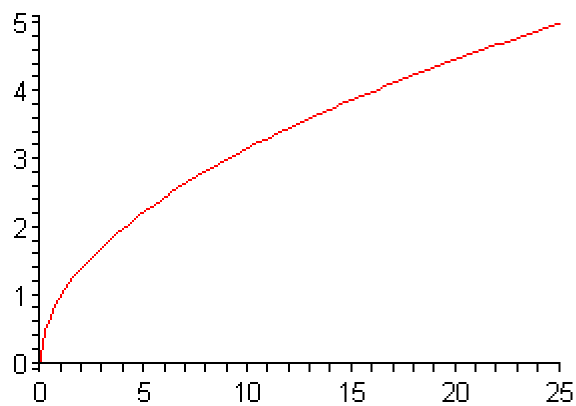
This has already been graphed once in this review, but this puts it here with all the other "important" graphs.



Example 10 Graph $y = \sqrt{x}$.

Solution

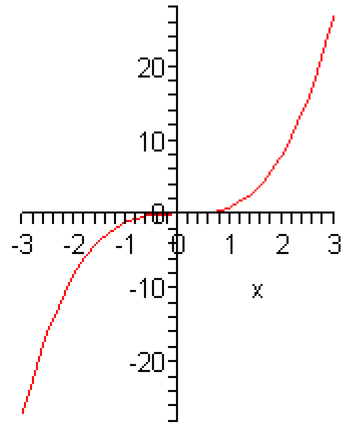
This one is fairly simple, we just need to make sure that we can graph it when need be.



Remember that the domain of the square root function is $x \geq 0$.

Example 11 Graph $y = x^3$

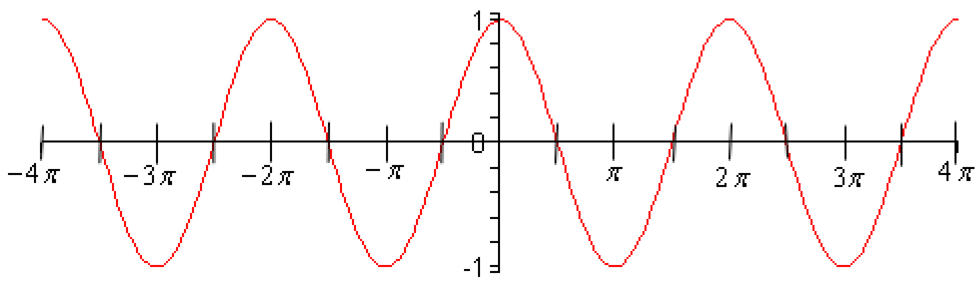
Again, there really isn't much to this other than to make sure it's been graphed somewhere so we can say we've done it.



Example 12 Graph $y = \cos(x)$

Solution

There really isn't a whole lot to this one. Here's the graph for $-4\pi \le x \le 4\pi$.



Let's also note here that we can put all values of *x* into cosine (which won't be the case for most of the trig functions) and so the domain is all real numbers. Also note that

$$-1 \le \cos(x) \le 1$$

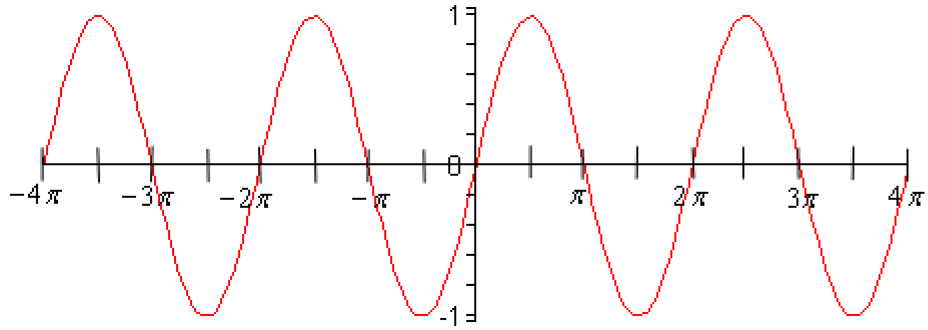
It is important to notice that cosine will never be larger than 1 or smaller than -1. This will be useful on occasion in a calculus class. In general we can say that

$$-R \le R\cos(\omega x) \le R$$

Example 13 Graph $y = \sin(x)$

Solution

As with the first problem in this section there really isn't a lot to do other than graph it. Here is the graph.



From this graph we can see that sine has the same range that cosine does. In general

$$-R \le R\sin(\omega x) \le R$$

As with cosine, sine itself will never be larger than 1 and never smaller than -1. Also the domain of sine is all real numbers.

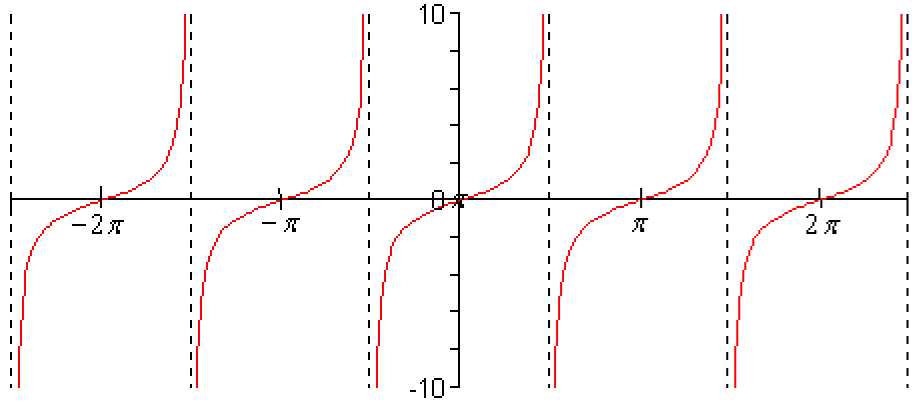
Example 14 Graph $y = \tan(x)$.

Solution

In the case of tangent we have to be careful when plugging x's in since tangent doesn't exist wherever cosine is zero (remember that $\tan x = \frac{\sin x}{\cos x}$). Tangent will not exist at

$$x = \cdots, -\frac{5\pi}{2}, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

and the graph will have asymptotes at these points. Here is the graph of tangent on the range $-\frac{5\pi}{2} < x < \frac{5\pi}{2}$.



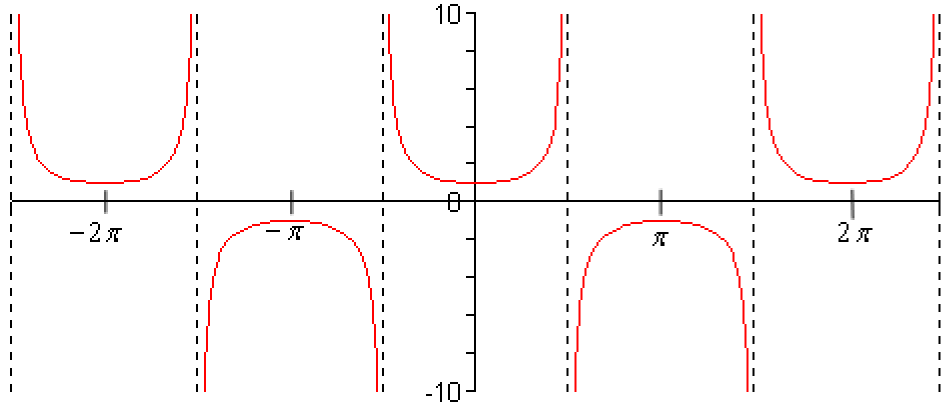
Example 15 Graph $y = \sec(x)$

Solution

As with tangent we will have to avoid x's for which cosine is zero (remember that $\sec x = \frac{1}{\cos x}$). Secant will not exist at

$$x = \cdots, -\frac{5\pi}{2}, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

and the graph will have asymptotes at these points. Here is the graph of secant on the range $-\frac{5\pi}{2} < x < \frac{5\pi}{2}$.



Notice that the graph is always greater than 1 and less than -1. This should not be terribly surprising. Recall that $-1 \le \cos(x) \le 1$. So, one divided by something less than one will be greater than 1. Also, $1/\pm 1 = \pm 1$ and so we get the following ranges for secant.

$$\sec(\omega x) \ge 1 \qquad \text{and} \qquad \sec(\omega x) \le -1$$

Limits

Introduction

In this chapter we will be looking at the first of three major topics that will be covered in this course. The topic that we will be examining in this chapter is that of Limits. While we will be spending the least time on limits in comparison to the other two topics limits are very important in the study of Calculus. We will be seeing limits in a variety of places once we move out of this chapter. In particular we will see that limits are part of the formal definition of the other two major topics.

Here is a quick listing of the material that will be covered in this chapter.

Tangent Lines and Rates of Change – In this section we will take a look at two problems that we will see time and again in this course. These problems will be used to introduce the topic of limits.

The Limit – Here we will take a conceptual look at limits and try to get a grasp on just what they are and what they can tell us.

One-Sided Limits – A brief introduction to one-sided limits.

Limit Properties – Properties of limits that we'll need to use in computing limits. We will also compute some basic limits in this section

Computing Limits – Many of the limits we'll be asked to compute will not be "simple" limits. In other words, we won't be able to just apply the properties and be done. In this section we will look at several types of limits that require some work before we can use the limit properties to compute them.

Limits Involving Infinity – Here we will take a look at limits that involve infinity. This includes limits that are infinity and limits at infinity. We'll also take a brief look at asymptotes.

Continuity – In this section we will introduce the concept of continuity and how it relates to limits. We will also see the Mean Value Theorem in this section.

The Definition of the Limit – We will give the exact definition of a limit in this section. This section is not always covered in a standard calculus class.

Tangent Lines and Rates of Change

In this section we are going to take a look at two fairly important problems in the study of calculus. There are two reasons for looking at these problems now.

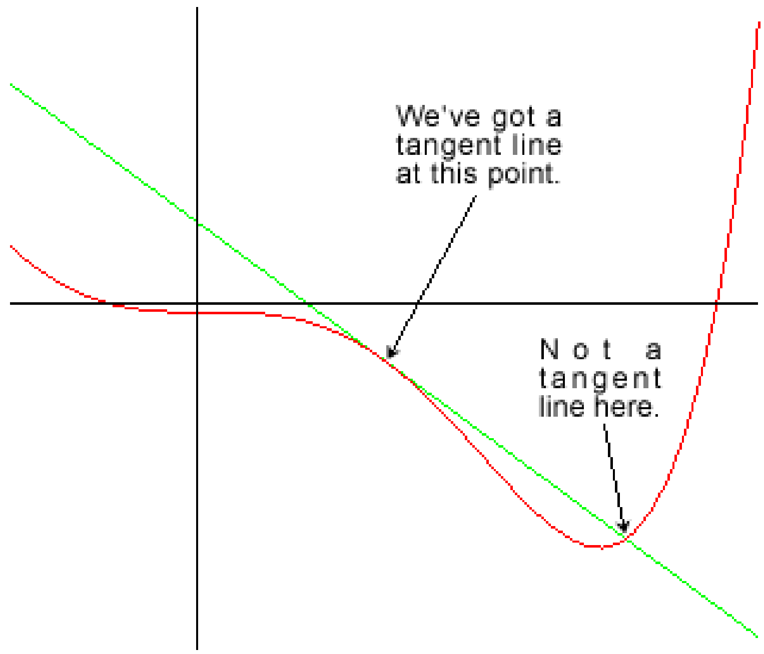
First, the second problem that we're going to be looking at is one of the most important concepts that we'll encounter in the second chapter of this course. In fact, it's probably one of the most important concepts that we'll encounter in the whole course. So looking at it now will get us to start thinking about it from the very beginning.

The second reason is that both of these problems will lead us into the study of limits, which is the topic of this chapter after all.

Tangent Lines

The first problem that we're going to take a look at is the tangent line problem. Before getting into this problem it would probably be best to define a tangent line.

A tangent line to the function $f(x)$ at the point $x=a$ is a line that just touches the graph of the function at the point in question and is also "parallel" to the graph at that point. Take a look at the graph below.



In this graph the line is a tangent line at the indicated point because it just touches the graph at that point and is also "parallel" to the graph at that point. Likewise, at the second point shown, the line does just touch the graph at that point, but it is not "parallel" to the graph at that point and so it's not a tangent line to the graph at that point.

At the second point shown (the point where the line isn't a tangent line) we will sometimes call the line a **secant line**.

We've used the word parallel a couple of times now and we should probably be a little careful with it. In general, we will think of a line and a graph as being parallel at a point if they are both moving in the same direction at that point. So, in the first point above the graph and the line are moving in the same direction and so we will say they are parallel at that point. At the second point, on the other hand, the line and the graph are not moving in the same direction and so they aren't parallel at that point.

Okay, now that we've gotten the definition of a tangent line out of the way let's move on to the tangent line problem. That's probably best done with an example.

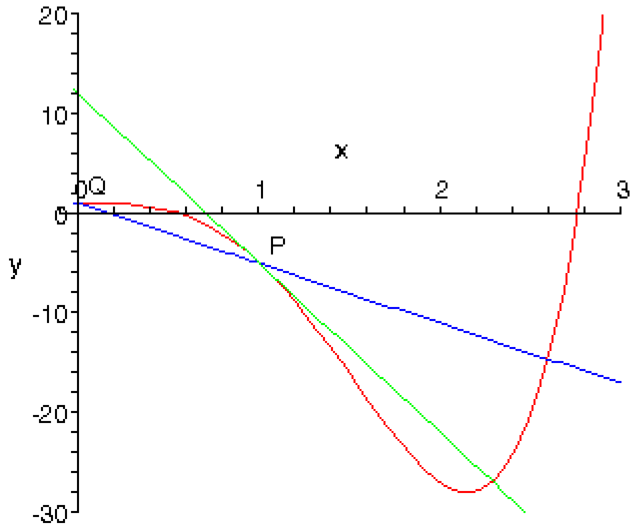
Example 1 Find the tangent line to $f(x) = x^5 - 8x^3 + x^2 + 1$ at $x = 1$.

Solution
We know from algebra that to find the equation of a line we need either two points or a single point and the slope of the line. Since we know that we are after a tangent line we do have a point that is on the tangent line. The tangent line and the graph of the function must touch at $x = 1$ so the point $(1, f(1)) = (1, -5)$ must be on the tangent line.

Now we reach the problem. This is all that we know about the tangent line. In order to find the tangent line we need either a second point or the slope of the tangent line. Since the only reason for needing a second point is to allow us to find the slope of the tangent line let's just concentrate on seeing if we can determine the slope of the tangent line.

At this point in time all that we're going to be able to do is to get an estimate for the slope of the tangent line, but if we do it correctly we should be able to get an estimate that is in fact the actual slope of the tangent line. We'll do this by starting with the point that we're after, let's call it $P = (1, -5)$. We will then pick another point that lies on the graph of the function, let's call that point $Q=(x, f(x))$.

For the sake of argument let's take choose $x=0$ and so the second point will be $(0, 1)$. Below is a graph of the function, the tangent line and the secant line that connects P and Q.

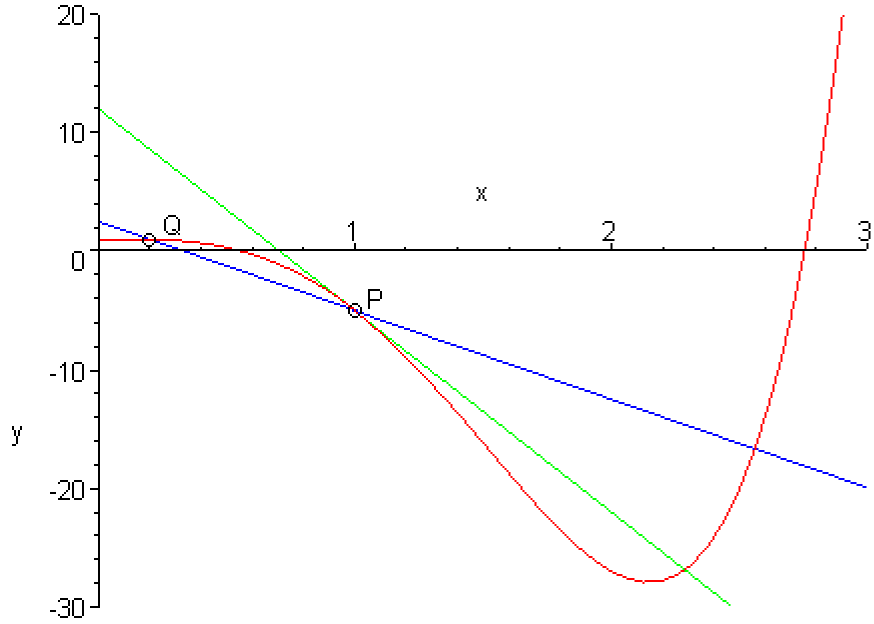


We can see from this graph that the two lines are somewhat similar and so the slope of the secant line should be somewhat close to the actual slope of the tangent line. The slope of the secant line is

$$m_{PQ} = \frac{f(0) - f(1)}{0-1} = \frac{1-(-5)}{-1} = -6$$

Now, if we weren't too interested in accuracy we could say this is good enough and use this as an estimate of the slope of the tangent line. However, we would like an estimate that is at least somewhat close the actual value. So, to get a better estimate we can take an x that is closer to $x=1$ and redo the work above to get a new estimate on the slope. We could then take a third value of x even closer yet and get an even better estimate.

In other words, as we take Q closer and closer to P the slope of the secant line connecting Q and P should be getting closer and closer to the slope of the tangent line. If you are viewing this on the web, the image below shows this process.



As you can see as we moved Q in closer and closer to P the secant lines does start to look more and more like the tangent line and so the slopes should be getting closer and closer.

In this figure we only looked at Q's that were to the left of P, but we could have just as easily used Q's that were to the right of P and we would have received the same results. In fact, we should always take a look at Q's that are on both sides of P. In this case the same thing is happening on both sides of P. However, we will see eventually that doesn't have to happen. Therefore we should always take a look at what is happening on both sides of the point in question.

So, let's see if we can come up with an estimation of the slope of the tangent line. In order to simplify the process let's get a formula for the slope of the line between P and Q, m_{PQ} that will work for any x that we choose to work with.

$$\begin{aligned} m_{PQ} &= \frac{f(x) - f(1)}{x-1} \\ &= \frac{x^5 - 8x^3 + x^2 + 1 - (-5)}{x-1} \\ &= \frac{x^5 - 8x^3 + x^2 + 6}{x-1} \end{aligned}$$

Now, let's pick some values of x getting closer and closer to $x=1$, plug in and get some slopes.

x	m_{PQ}	x	m_{PQ}
0	-6	2	-22

0.5	-10.5625	1.5	-23.3125
0.9	-15.6849	1.1	-18.2749
0.99	-16.869805	1.01	-17.129795
0.999	-16.986998	1.001	-17.012998
0.9999	-16.998799	1.0001	-17.001299
0.99999	-16.999869	1.00001	-17.000129

So, if we take x's to the left of 1 and move them in very close to 1 it appears that the slope of the secant lines appears to be approaching -17. Likewise, if we take x's to the right of 1 and move them in very close to 1 the slope of the secant lines again appears to be approaching -17.

Based on this evidence it seems that the slopes of the secant lines are approaching -17 as we move in towards $x=1$, so we will estimate that the slope of the tangent line is also -17. In fact, this is the correct value as we will be able to prove eventually.

Now, the equation of the line that goes through $(a, f(a))$ is given by

$$y = f(a) + m(x-a).$$

Therefore, the slope of the tangent line to $f(x) = x^5 - 8x^3 + x^2 + 1$ at $x = 1$ is

$$y = -5 - 17(x-1) = -17x + 12$$

There are a couple of important points to note about our work above. First, we looked at points that were on both sides of $x=1$. In this kind of process it is important to never assume that what is happening on one side of a point will also be happening on the other side as well. We should always look at what is happening on both sides of the point. In this example we could sketch a graph and from that guess that what is happening on one side will also be happening on the other, but we will usually not have the graphs in front of us.

Next, notice that when we say we're going to move in close to the point in question we do mean that we're going to move in very close and we also used more than just a couple of points. We should never try to determine a trend based on a couple of points that aren't really all that close to the point in question.

Last, we were after something that was happening at $x=1$ and we couldn't actually plug $x=1$ into our formula for the slope. Despite this limitation we were able to determine some information about what was happening at $x=1$ simply by looking at what was happening around $x=1$. This is more important than you might at first realize. We will discuss this point in detail in later sections.

Rates of Change

The next problem that we need to look at is the rate of change problem. This will turn out to be one of the most important concepts that we will look at throughout this course.

Here we are going to consider a function, *f(x)*, that represents some quantity that varies as *x* varies. For instance, maybe *f(x)* represents the amount of water in a holding tank after *x* minutes. Or maybe *f(x)* is the distance traveled by a car after *x* hours. In both of these example we used *x* to represent time. Of course *x* doesn't have to represent time, but it makes for examples that are easy to visualize.

What we want to do here is determine just how fast *f(x)* is changing at some point, say $x=a$. This is called the **instantaneous rate of change** or just **rate of change** of *f(x)* at $x=a$.

As with the tangent line problem all that we're going to be able to do at this point is to estimate the rate of change. So let's continue with the examples above and think of *f(x)* as something that is changing in time and *x* being the time measurement. Again *x* doesn't have to represent time but it will make the explanation a little easier. While we can't compute the instantaneous rate of change at this point we can find the average rate of change.

To compute the average rate of change of *f(x)* at $x=a$ all we need to do is to choose another point, *x*, and then the average rate of change will be,

$$\begin{aligned} A.R.C. &= \frac{\text{change in } f\left(x\right)}{\text{change in } x} \\ &= \frac{f\left(x\right)-f\left(a\right)}{x-a} \end{aligned}$$

Then to estimate the instantaneous rate of change at $x=a$ all we need to do is to choose values of *x* getting closer and closer to $x=a$ (don't forget to chose them on both sides of $x=a$) and compute values of *A.R.C.* We can then estimate the instantaneous rate of change form that.

Let's take a look at an example.

Example 2 Suppose that the amount of air in a balloon after *t* hours is given by

$$V\left(t\right)=t^3-6t^2+35$$

Estimate the instantaneous rate of change of the volume after 5 hours.

Solution

Okay. The first thing that we need to do is get a formula for the average rate of change of the volume. In this case this is,

$$\begin{aligned} A.R.C. &= \frac{V\left(t\right)-V\left(5\right)}{t-5} \\ &= \frac{t^3-6t^2+35-10}{t-5} \\ &= \frac{t^3-6t^2+25}{t-5} \end{aligned}$$

To estimate the instantaneous rate of change of the volume at $t=5$ we just need to pick values of t that are getting closer and closer to $t=5$. Here is a table of values of t and the average rate of change for those values.

t	***A.R.C.***	***t***	***A.R.C.***
6	25.0	4	7.0
5.5	19.75	4.5	10.75
5.1	15.91	4.9	14.11
5.01	15.0901	4.99	14.9101
5.001	15.009001	4.999	14.991001
5.0001	15.00090001	4.9999	14.99910001

So, from this table it looks like the average rate of change is approaching 15 and so we can estimate that the instantaneous rate of change is 15 at this point.

So, just what does this tell us about the volume at this point? Let's put some units on the answer from above. This might help us to see what is happening to the volume at this point. Let's suppose that the units on the volume were in cm^3. The units on the rate of change (both average and instantaneous) are then cm^3/hr.

We have estimated that the volume is changing at a rate of 15 cm^3/hr. This means that at $t=5$ the volume is changing in such a way that, if the rate were constant, then an hour later there would be 15 cm^3 more air in the balloon than there was at $t=5$.

We do need to be careful here however. In reality there probably won't be 15 cm^3 more air in the balloon after an hour. The rate at which the volume is changing is not constant and so we can't make any real determination as to what the volume will be in another hour. What we can say is that the volume is increasing, since the instantaneous rate of change is positive, and if we had rates of change for other values of t we could compare the numbers and see if the rate of change is faster or slower at the other points.

For instance, at $t=4$ the instantaneous rate of change is 0 cm^3/hr and at $t=3$ the instantaneous rate of change is -9 cm^3/hr. I'll leave it to you to check these rates of change. In fact, that would be a good exercise to see if you can build a table of values that will support my claims on these rates of change.

Anyway, back to the example. At $t=4$ the rate of change is zero and so at this point in time the volume is not changing at all. That doesn't mean that it will not change in the future. It just means that exactly at $t=4$ the volume isn't changing. Likewise at $t=3$ the volume is decreasing since the rate of change at that point is negative. We can also say that, regardless of the increasing/decreasing aspects of the rate of change, the volume of the balloon is changing faster at $t=4$ than it is at $t=3$ since 15 is larger than 9.

We will be talking a lot more about rates of change when we get into the next chapter.

Velocity Problem
Let's briefly look at the velocity problem. Many calculus books will treat this as its own problem. I however, like to think of this as a special case of the rate of change problem. In the velocity problem we are given a position function of an object, *f(t),* that gives the position of an object at time *t*. Then to compute the instantaneous velocity of the object we just need to recall that the velocity is nothing more than the rate at which the position is changing.

In other words, to estimate the instantaneous velocity we would first compute the average velocity,

$$\begin{aligned} A.V. &= \frac{\text{change in position}}{\text{time traveled}} \\ &= \frac{f\left(t\right)-f\left(a\right)}{t-a} \end{aligned}$$

and then take values of *t* closer and closer to *t=a* and use these values to estimate the instantaneous velocity.

Change of Notation
There is one last thing that we need to do in this section before we move on. The main point of this section was to introduce us to a couple of key concepts and ideas that we will see through out this course as well as get us started down the path towards limits.

Before we move into limits officially let's go back and do a little work that will relate both (or all three if you include velocity as a separate problem) problems to a more general concept.

First, notice that whether we wanted the tangent line, instantaneous rate of change, or instantaneous velocity each of these came down to using exactly the same formula. Namely,

$$\frac{f\left(x\right)-f\left(a\right)}{x-a} \tag{1}$$

This should suggest that all three of these problems are then really the same problem. In fact this is the case as we will see in the next chapter. We are really working the same problem in each of these cases the only difference is the interpretation of the results.

In preparation for the next section where we will discuss this in much more detail we need to do a quick change of notation. It's easier to do here since we've already invested a fair amount of time into these problems.

In all of these problems we wanted to determine what was happening at *x=a*. To do this we chose another value of *x* and plugged into (1). For what we were doing here that is probably most intuitive way of doing it. However, when we start looking at these problems as a single problem (1) will not be the best formula to work with.

What we'll do instead is to first determine how far from $x=a$ we want to move and then define our new point based on that decision. So, if we want to move a distance of h from $x=a$ the new point would be $x=a+h$.

As we saw in our work above it is important to take values of x that are both sides of $x=a$. This way of choosing new value of x will do this for us. If $h>0$ we will get value of x that are to the right of $x=a$ and if $h<0$ we will get values of x that are to the left of $x=a$.

Now, with this new way of getting a second x, (1) will become,

$$\frac{f(x)-f(a)}{x-a}=\frac{f(a+h)-f(a)}{a+h-a}=\frac{f(a+h)-f(a)}{h} \qquad \textbf{(2)}$$

On the surface it might seem that (2) is going to be an overly complicated way of dealing with this stuff. However, as we will see it will often be easier to deal with (2) than it will be to deal with (1).

The Limit

In the previous section we looked at a couple of problems and in both problems we had a function (slope in the tangent problem case and average rate of change in the rate of change problem) and we wanted to know how that function was behaving at some point $x=a$. At this stage of the game we no longer care where the functions came from and we no longer care if we're going to see them down the road again or not. All that we need to know or worry about is that we've got these functions and we want to know something about them.

To answer the question in the last section we choose values of x that got closer and closer to $x=a$ and we plugged these into the function. We also made sure that we looked at values of x that were on both the left and the right of $x=a$. Once we did this we looked at our table of function values and saw what the function values were approaching as x got closer and closer to $x=a$ and used this to guess the value that we were after.

This process is called **taking a limit** and we have some notation for this. The limit notation for the two problems from the last section is,

$$\lim_{x\to 1}\frac{x^5-8x^3+x^2+6}{x-1}=-17$$

$$\lim_{t\to 5}\frac{t^3-6t^2+25}{t-5}=15$$

In this notation we will note that we always give the function that we're working with and we also give the value of x (or t) that we are moving in towards.

In this section we are going to take an intuitive approach to limits and try to get a feel for what they are and what they can tell us about a function. With that goal in mind we are not going to get into how we actually compute limits yet. We will instead rely on what we did in the previous section as well as another approach to guess the value of the limits.

Both of the approaches that we are going to use in this section are designed to help us understand just what limits are. In general we don't typically use the methods in this section to compute limits. We will look at actually computing limits in a couple of sections.

Let's first start off with the following "definition" of a limit.

Definition

We say that the limit of *f(x)* is *L* as *x* approaches *a* and write this as

$$\lim_{x \to a} f\left(x\right) = L$$

provided we can make *f(x)* as close to *L* as we want for all *x* sufficiently close to *a*, from both sides, without actually letting *x* be *a*.

This is not the exact, precise definition of a limit. If you would like to see the exact definition of a limit you should check out the last section in this chapter. The definition given above is more of a "working" definition. This definition helps us to get an idea of just what limits are and what they can tell us about functions.

So just what does this definition mean? Well let's suppose that we know that the limit does in fact exist. According to our definition we can then decide how close to *L* that we'd like to make *f(x)*. For sake of argument let's suppose that we want to make *f(x)* no more that 0.001 away from *L*. This means that we want one of the following

$$f\left(x\right) - L < 0.001 \qquad \text{if } f\left(x\right) \text{ is larger than L}$$

$$L - f\left(x\right) < 0.001 \qquad \text{if } f\left(x\right) \text{ is smaller than L}$$

Now according to the definition this means that if we get *x* sufficiently close to we can make one of the above true. However, it actually says a little more. It actually says that somewhere out there in the world is a value of *x*, say *X*, so that for all *x's* that are closer to *a* than *X* then one of the above statements will be true.

In somewhat simpler terms the definition says that as *x* gets closer and closer to *x=a* (from both sides of course…) then *f(x)* **must** be getting closer and closer to *L*. Or, as we move in towards *x=a* then *f(x)* **must** be moving in towards *L*.

It is important to note once again that we must look at values of *x* that are on both sides of *x=a*. We should also note that we are not allowed to use *x=a* in the definition. We will often use the information that limits give us to get some information about what is going on right at *x=a*, but the limit itself is not concerned with what is actually going on at *x=a*. The limit is only concerned with what is going on around the point *x=a*. This is an important concept about limits that we need to keep in mind.

An alternative notation that we will occasionally use in denoting limits is

$$f(x) \to L \qquad \text{as} \qquad x \to a$$

How do we use this definition to help us estimate limits? We do exactly what we did in the previous section. We take *x's* on both sides of $x=a$ that move in closer and closer to a and we plug these into our function. We then look to see if we can determine what number the function values are moving in towards and use this as our estimate.

Let's work an example.

Example 1 Estimate the value of the following limit.

$$\lim_{x\to 2} \frac{x^2+4x-12}{x^2-2x}$$

Solution

Notice that I did say estimate the value of the limit. Again, we are not going to directly compute limits in this section. The point of this section is to give us a better idea of how limits work and what they can tell us about the function.

So, with that in mind we are going to work this in pretty much the same way that we did in the last section. We will choose values of x that get closer and closer to $x=2$ and plug these values into the function. Doing this gives the following table of values.

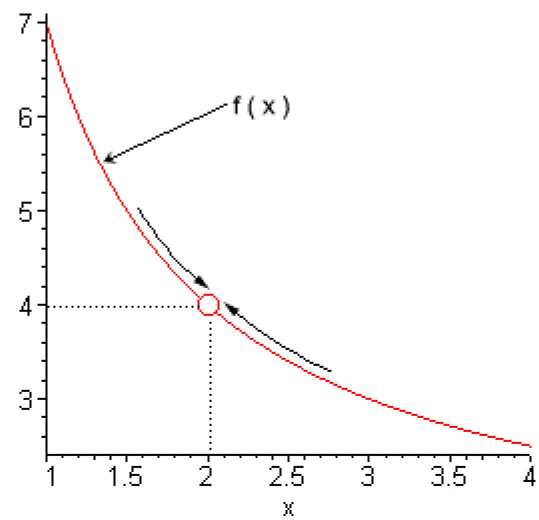
x	$f(x)$	x	$f(x)$
2.5	3.4	1.5	5.0
2.1	3.857142857	1.9	4.157894737
2.01	3.985074627	1.99	4.015075377
2.001	3.998500750	1.999	4.001500750
2.0001	3.999850007	1.9999	4.000150008
2.00001	3.999985000	1.99999	4.000015000

Note that we made sure and picked values of x that were on both sides of $x=2$ and that we moved in very close to $x=2$ to make sure that any trends that we might be seeing are in fact correct. Also notice that we can't actually plug in $x=2$ into the function as this would give us a division by zero error. This is not a problem since the limit doesn't care what is happening at the point in question.

From this table it appears that the function is going to 4 as x approaches 2, so

$$\lim_{x\to 2} \frac{x^2+4x-12}{x^2-2x} = 4.$$

Let's think a little bit more about what's going on here. Let's graph the function from the last example. The graph of the function in the range of x's that were interested in is shown below.



First, notice that there is a rather large open dot at x=2. This is there to remind us that the function (and hence the graph) doesn't exist at x=2.

As we where plugging in values of x into the function we are in effect moving along the graph in towards the point as $x=2$. This is shown in the graph above by the two arrows on the graph.

When we are computing limits the question that we are really asking is what y value is our graph approaching as we move in towards $x=2$ on our graph. We are **NOT** asking what y value the graph takes at the point in question. In other words, we are asking what the graph is doing **around** the point $x=2$. Again, in our case we can see that as x moves in towards 2 (from both sides) the function is approaching $y=4$. Therefore we can say that the limit is in fact 4.

So what have we learned about limits? Limits are asking what the function is doing **around** $x=a$ and are **not** concerned with what the function is actually doing at $x=a$. This is a good thing as many of the functions that we'll be looking at won't even exist at $x=a$ as in our last example.

Let's work another example to drive this point home.

Example 2 Estimate the value of the following limit.

$$\lim_{x \to 2} g(x) \qquad \text{where,} \qquad g(x) = \begin{cases} \dfrac{x^2+4x-12}{x^2-2x} & \text{if } x \neq 2 \\ 5 & \text{if } x = 2 \end{cases}$$

Solution

The first thing to note here is that this is exactly the same function as the first example with the exception that we've now given it a value for $x=2$. So, let's first note that

$$g(2) = 5$$

As far as estimating the value of this limit goes, nothing has changed in comparison to the first example. We could build up a table of values as we did in the first example or we could take a quick look at the graph of the function. Either method will give us the value of the limit.

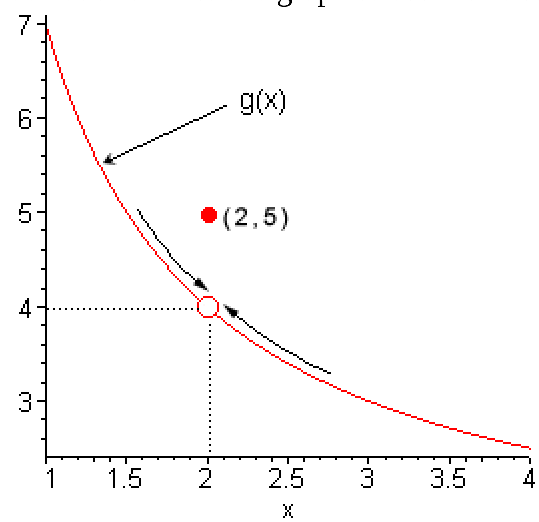
Lets' first take a look at a table of values and see what that tells us. Notice that the presence of the value for the function at $x=2$ will not change our choices for x. We only choose values of x that are getting closer to $x=2$ but we never take $x=2$. In other words the table of values that we used in the first example will be exactly the same table that we'll use here. So, since we've already got it down once there is no reason to redo it here.

From this table it is again clear that the limit is,

$$\lim_{x\to 2} g\left(x\right) = 4$$

The limit is **NOT** 5! Remember from the discussion after the first example that limits do not care what the function is actually doing at the point in question. Limits are only concerned with what is going on **around** the point. Since the only thing about the function that we actually changed was its behavior at $x=2$ this will not change the limit.

Let's also take a quick look at this functions graph to see if this says the same thing.



Again, we can see that as we move in towards $x=2$ on our graph the function is still approaching a y value of 4. Remember that we are only asking what the function is doing **around** $x=2$ and we don't care what the function is actually doing at $x=2$. The graph then also supports the conclusion that the limit is,

$$\lim_{x\to 2} g\left(x\right) = 4$$

Let's make the point one more time just to make sure we've got it. Limits are **not** concerned with what is going on at $x=a$. Limits are only concerned with what is going on **around** $x=a$. We keep saying this, but it is a very important concept about limits that we must always keep in mind. So, we will take every opportunity to remind ourselves of this idea.

Since limits aren't concerned with what is actually happening at $x=a$ we will, on occasion, see situations like the previous example where the limit at a point and the function value at a point are different. This won't always happen of course. There are times where the function value and the limit at a point are the same and we will eventually see some examples of those. It is important however, to not get excited about things when the function and the limit do not take the same value at a point. It happens sometimes and so we will need to be able to deal with those cases when they arise.

Let's take a look another example to try and beat this idea into the ground.

Example 3 Estimate the value of the following limit.

$$\lim_{\theta \to 0} \frac{1-\cos(\theta)}{\theta}$$

Solution

First don't get excited about the θ in function. It's just a letter, just like x is a letter! It's a Greek letter, but it's a letter and you will be asked to deal with Greek letters on occasion so it's a good idea to start getting used to them at this point.

Now, also notice that if we plug in $\theta=0$ that we will get division by zero and so the function doesn't exist at this point. Actually, we get 0/0 at this point, but because of the division by zero this function does not exist at $\theta=0$.

So, as we did in the first example let's get a table of values and see what if we can guess what value the function is heading in towards.

θ	$f(\theta)$	θ	$f(\theta)$
1	0.45969769	-1	-0.45969769
0.1	0.04995835	-0.1	-0.04995835
0.01	0.00499996	-0.01	-0.00499996
0.001	0.00049999	-0.001	-0.00049999

Okay, it looks like the function is moving in towards a value of zero as θ moves in towards 0, from both sides of course.

Therefore, the we will guess that the limit has the value,

$$\lim_{\theta \to 0} \frac{1-\cos(\theta)}{\theta} = 0$$

So, once again, the limit had a value even though the function didn't exist at the point we were interested in.

It's now time to work a couple of more examples that will lead us into the next idea about limits that we're going to want to discuss.

Example 4 Estimate the value of the following limit.

$$\lim_{t\to 0} \cos\left(\frac{\pi}{t}\right)$$

Solution

Let's build up a table of values and see what's going on with our function in this case.

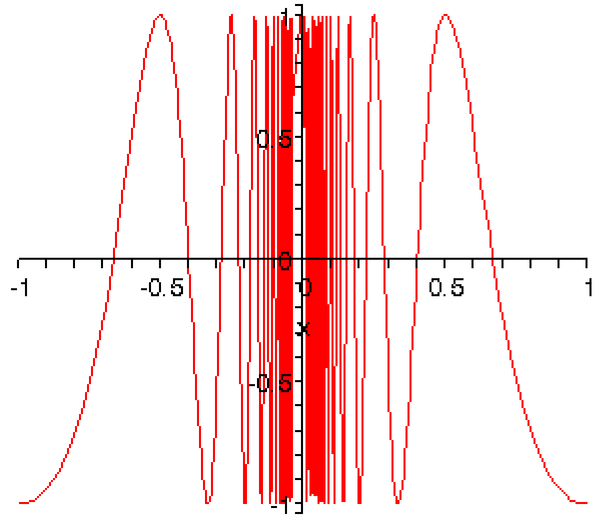
t	*f(t)*	*t*	*f(t)*
1	-1	-1	-1
0.1	1	-0.1	1
0.01	1	-0.01	1
0.001	1	-0.001	1

Now, if we were to guess the limit from this table we would guess that the limit is 1. However, if we did make this guess we would be wrong. Consider any of the following function evaluations.

$$f\left(\frac{1}{2001}\right) = -1 \qquad f\left(\frac{2}{2001}\right) = 0 \qquad f\left(\frac{4}{4001}\right) = \frac{\sqrt{2}}{2}$$

In all three of these function evaluations we evaluated the function at a number that is less that 0.001 and got three totally different numbers. Recall that the definition of the limit that we're working with requires that the function be approaching a single value (our guess) as *t* gets closer and closer to the point in question. It doesn't say that only some of the function values must be getting closer to the guess. It says that all the function values must be getting closer and closer to our guess.

To see what's happening here a graph of the function would be convenient.



From this graph we can see that as we move in towards $t=0$ the function starts oscillating wildly and in fact the oscillations increases in speed the closer to $t=0$ that we get. Recall from our definition of the limit that in order for a limit to exist the function must be settling down in towards a single value as we get closer to the point in question.

This function clearly does not settle in towards a single number and so this limit does not exist!

This last example points out the drawback of just picking values of x using a table of function values to estimate the value of a limit. The values of x that we chose in the previous example were valid and in fact were probably values that many would have picked. In fact they were exactly the same values we used in the problem before this one and they worked in that problem!

When using a table of values there will always be the possibility that we aren't choosing the correct values and that we will guess incorrectly for our limit. This is something that we should always keep in mind when doing this to guess the value of limits. In fact, this is such a problem that after this section we will never use a table of values to guess the value of a limit again.

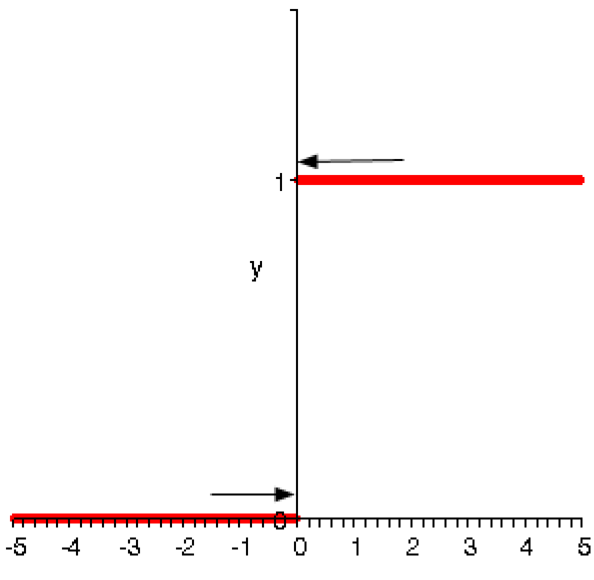
This last example also has shown us that limits do not have to exist. To this point we've only seen limits that have existed, but that just doesn't always have to be the case.

Let's take a look at one more example in this section.

Example 5 Estimate the value of the following limit.

$$\lim_{t \to 0} H(t) \qquad \text{where,} \qquad H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

Solution
This function is often called either the **Heaviside** or **step** function. We could use a table of values to estimate the limit, but it's probably just as quick in this case to use the graph so let's do that. Below is the graph of this function.



We can see from the graph that if we approach $t=0$ from the right side the function is moving in towards a y value of 1. Well actually it's just staying at 1, but in the terminology that we've been using in this section it's moving in towards 1…

Also, if we move in towards $t=0$ from the left the function is moving in towards a y value of 0.

According to our definition of the limit the function needs to move in towards a single value as we move in towards $t=a$ (from both sides). This isn't happening in this case and so in this example we will also say that the limit doesn't exist.

Let's summarize what we (hopefully) learned in this section. In the first three examples we saw that limits do not care what the function is actually doing at the point in question. They only are concerned with what is happening around the point. In fact, we can have limits at $x=a$ even if the function itself does not exist at that point. Likewise, even if a function exists at a point there is no reason (at this point) to think that the limit will have the same value as the function at that point. Sometimes the limit and the function will have the same value at a point and other times they won't have the same value.

Next, in the third and fourth examples we saw the main reason for not using a table of values to guess the value of a limit. In those examples we used exactly the same set of values, however they only worked in one of the examples. Using tables of values to guess the value of limits is simply not a good way to get the value of a limit. This is the only section in which we will do this. Tables of values should always be your last choice in finding values of limits.

The last two examples showed us that not all limits will in fact exist. We should not get locked into the idea that limits will always exist. In most calculus courses we work with limits that almost always exist and so it's easy to start thinking that limits always exist. Limits don't always exist and so don't get into the habit of assuming that they will.

Finally, we saw in the fourth example that the only way to deal with the limit was to graph the function. Sometimes this is the only way, however this example also illustrated the drawback of using graphs. In order to use a graph to guess the value of the limit you need to be able to actually sketch the graph. For many functions this is not that easy to do.

There is another drawback in using graphs. Even if you actually have the graph it's only going to be useful if the *y* value is approaching an integer. If the *y* value is approaching say $\frac{-15}{123}$ there is no way that you're going to be able to guess that value from the graph and we are usually going to want exact values for our limits.

So while graphs of functions can, on occasion, make your life easier in guessing values of limits they are again probably not the best way to get values of limits. They are only going to be useful if you can get your hands on it and the value of the limit is a "nice" number.

The natural question then is why did we even talk about using tables and/or graphs to estimate limits if they aren't the best way. There were a couple of reasons.

First, they can help us get a better understanding of what limits are and what they can tell us. If we don't do at least a couple of limits in this way we might not get all that good of an idea on just what limits are.

The second reason for doing limits in this way is to point out their drawback so that we aren't tempted to use them all the time!

We will eventually talk about how we really do limits. However, there is one more topic that we need to discuss before doing that. Since this section has already gone on for a while we will talk about this in the next section.

One-Sided Limits

In the final two examples in the previous section we saw two limits that did not exist. However, the reason for each of the limits not existing was different for each of the examples.

We saw that

$$\lim_{t \to 0} \cos\left(\frac{\pi}{t}\right)$$

did not exist because the function did not settle down to a single value as *t* approached *t=0*. The closer to *t=0* we moved the more wildly the function oscillated and in order for a limit to exist the function must settle down to a single value.

However we saw that

$$\lim_{t \to 0} H\left(t\right) \qquad \text{where,} \qquad H\left(t\right) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

did not exist not because the function didn't settle down to a single number as we moved in towards *t=0*, but instead because it settled into two different numbers depending on which side of *t=0* we were on.

In this case the function was a very well behaved function, unlike the first function. The only problem was that, as we approached *t=0*, the function was moving in towards different numbers on each side. We would like a way to differentiate between these two examples.

We do this with **one-sided limits**. As the name implies, with one-sided limits we will only be looking at one side of the point in question. Here are the definitions for the two one sided limits.

Right-handed limit

We say

$$\lim_{x \to a^+} f\left(x\right) = L$$

provided we can make *f(x)* as close to *L* as we want for all *x* sufficiently close to *a* and *x>a* without actually letting *x* be *a*.

Left-handed limit

We say

$$\lim_{x \to a^-} f\left(x\right) = L$$

provided we can make *f(x)* as close to *L* as we want for all *x* sufficiently close to *a* and *x<a* without actually letting *x* be *a*.

Note that the change in notation is very minor and in fact might be missed if you aren't paying attention. The only difference is the bit that is under the "lim" part of the limit. For the right-handed limit we now have $x \to a^+$ (note the "+") which means that we know will only look at *x>a*. Likewise for the left-handed limit we have $x \to a^-$ (note the "-") which means that we will only be looking at *x<a*.

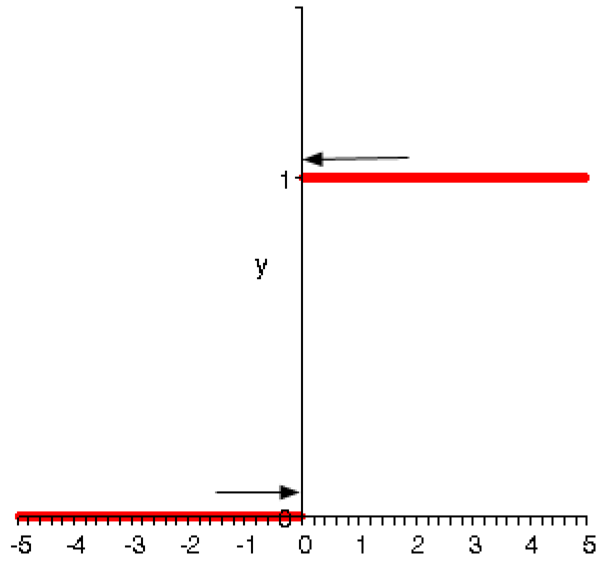
So when we are looking at limits it's now important to pay very close attention to see whether we are doing a normal limit or one of the one-sided limits. Let's now take a look at the some of the problems from the last section and look at one-sided limits instead of the normal limit.

Example 1 Estimate the value of the following limits.

$$\lim_{t \to 0^+} H(t) \qquad \text{and} \qquad \lim_{t \to 0^-} H(t) \qquad \text{where, } H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

Solution

Here's the graph of this function.



So, we can see that if we stay to the right of $t=0$ (*i.e.* $t>0$) then the function is moving in towards a value of 1 as we get closer and closer to $t=0$. We can therefore say that,

$$\lim_{t \to 0^+} H(t) = 1$$

Likewise, if we stay to the left of $t=0$ (*i.e* $t<0$) the function is moving in towards a value of 0 as we get closer and closer to $t=0$. Therefore,

$$\lim_{t \to 0^-} H(t) = 0$$

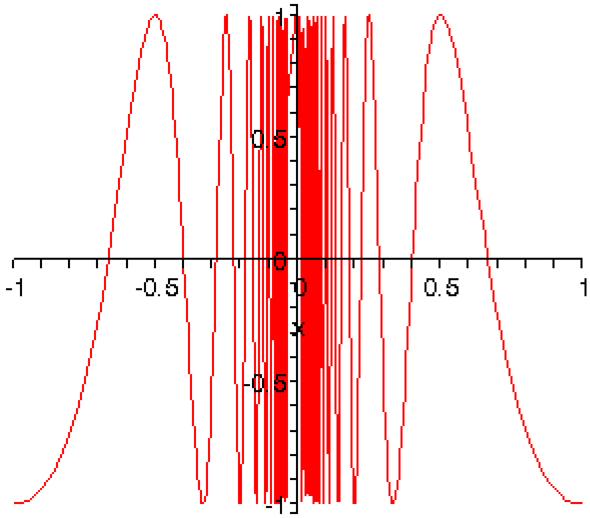
In this example we do get one-sided limits even though the normal limit itself doesn't exist.

Example 2 Estimate the value of the following limits.

$$\lim_{t \to 0^+} \cos\left(\frac{\pi}{t}\right) \qquad \lim_{t \to 0^-} \cos\left(\frac{\pi}{t}\right)$$

Solution

From the graph of this function shown below,



we can see that both of the one-sided limits suffer the same problem that the normal limit did. The function does not settle down to a single number on either side of *t=0*. Therefore, neither the left-handed nor the right-handed limit will exist.

So, one-sided limits don't have to exist just as normal limits aren't guaranteed to exist.

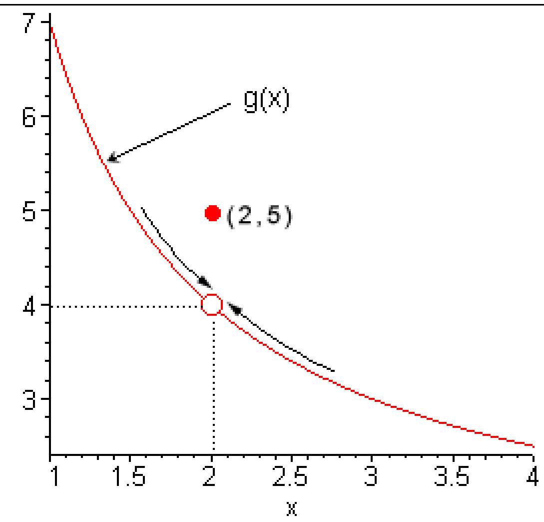
Let's take a look at another example from the previous section.

Example 3 Estimate the value of the following limits.

$$\lim_{x \to 2^+} g(x) \quad \text{and} \quad \lim_{x \to 2^-} g(x) \quad \text{where,} \quad g(x) = \begin{cases} \dfrac{x^2+4x-12}{x^2-2x} & \text{if } x \neq 2 \\ 5 & \text{if } x = 2 \end{cases}$$

Solution

So as we've done with the previous two examples, let's remind ourselves of the graph of this function.



In this case regardless of which side of *x=2* we are on the function is always approaching a value of 4 and so we get,

$$\lim_{x \to 2^+} g(x) = 4 \qquad \lim_{x \to 2^-} g(x) = 4$$

Note that one-sided limits do not care about what's happening at the point any more than normal limits do. They are still only concerned with what is going on around the point. The only real difference between one-sided limits and normal limits is the range of *x*'s that we look at when determining the value of the limit.

Now let's take a look at the first and last example in this section to get a very nice fact about the relationship between one-sided limits and normal limits. In the last example the one-sided limits as well as the normal limit existed and all three had a value of 4. In the first example the two one-sided limits both existed, but did not have the same value and the normal limit did not exist.

The relationship between one-sided limits and normal limits can be summarized by the following fact.

Fact

Given a function *f(x)* if,

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$$

then the normal limit will exist and

$$\lim_{x \to a} f(x) = L$$

Likewise, if

$$\lim_{x \to a} f(x) = L$$

then,

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$$

This fact can be turned around to also say that if

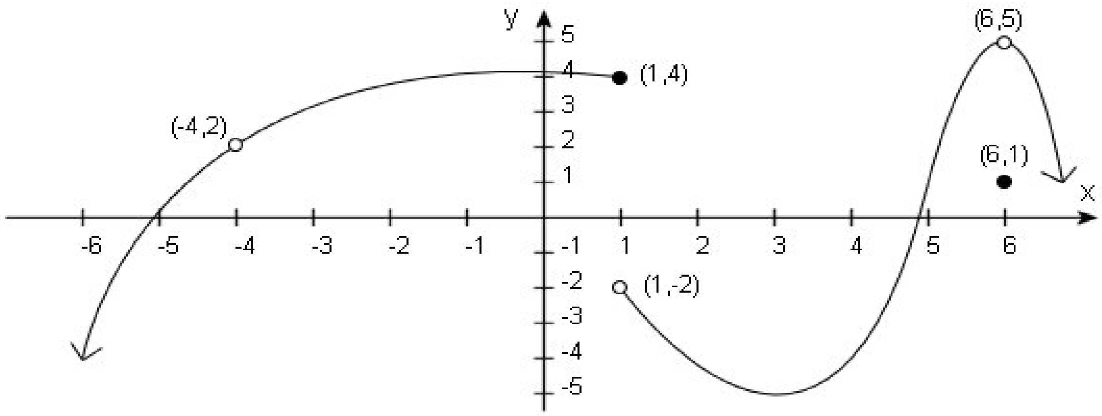
$$\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$$

then the normal limit will not exist.

This should make some sense. If the normal limit did exist then by the fact the two one-sided limits would have to exist and have the same value. So, if the two one-sided limits have different values (or don't even exist) then the normal limit simply can't exist.

Let's take a look at one more example to make sure that we've got all the ideas about limits down that we've looked at in the last couple of sections.

Example 4 Given the following graph,



compute each of the following.

(a) $f(-4)$ **(b)** $\lim_{x \to -4^-} f(x)$ **(c)** $\lim_{x \to -4^+} f(x)$ **(d)** $\lim_{x \to -4} f(x)$

(e) $f(1)$ **(f)** $\lim_{x \to 1^-} f(x)$ **(g)** $\lim_{x \to 1^+} f(x)$ **(h)** $\lim_{x \to 1} f(x)$

(i) $f(6)$ **(j)** $\lim_{x \to 6^-} f(x)$ **(k)** $\lim_{x \to 6^+} f(x)$ **(l)** $\lim_{x \to 6} f(x)$

Solution

(a) *f(-4)* doesn't exist. There is no closed dot for this value of *x* and so the function doesn't exist at this point.

(b) $\lim_{x \to -4^-} f(x) = 2$ The function is approaching a value of 2 as *x* moves in towards -4 from the left.

(c) $\lim_{x \to -4^+} f(x) = 2$ The function is approaching a value of 2 as *x* moves in towards -4 from the right.

(d) $\lim_{x \to -4} f(x) = 2$ We can do this one of two ways. Either we can use the fact here and notice that the two one-sided limits are the same and so the normal limit must exist and have the same value as the one-sided limits or just get the answer from the graph.

Also recall that a limit can exist at a point even if the function doesn't exist at that point.

(e) *f(1)=4*. The function will take on the *y* value where the closed dot is.

(f) $\displaystyle\lim_{x \to 1^-} f(x) = 4$ The function is approaching a value of 4 as *x* moves in towards 1 from the left.

(g) $\displaystyle\lim_{x \to 1^+} f(x) = -2$ The function is approaching a value of -2 as *x* moves in towards 1 from the right.

(h) $\displaystyle\lim_{x \to 1} f(x)$ doesn't exist. The two one-sided limits both exist, however they are different and so the normal limit doesn't exist.

(i) *f(6)=1*. The function will take on the *y* value where the closed dot is.

(j) $\displaystyle\lim_{x \to 6^-} f(x) = 5$ The function is approaching a value of 5 as *x* moves in towards 6 from the left.

(k) $\displaystyle\lim_{x \to 6^+} f(x) = 5$ The function is approaching a value of 5 as *x* moves in towards 6 from the right.

(l) $\displaystyle\lim_{x \to 6} f(x) = 5$ Again, we can use either the graph or the fact to get this. Also, once more remember that the limit doesn't care what is happening at the point and so it's possible for the limit to have a different value than the function at a point.

Hopefully over the last couple of sections you've gotten an idea on how limits work and what they can tell us about functions. Some of these ideas will be important in later sections so it's important that you have a good grasp on them.

Limit Properties

The time has almost come for us to actually compute some limits. However, before we do that we will need some properties of limits that will make our life somewhat easier.

Properties

First we will assume that $\displaystyle\lim_{x \to a} f(x)$ and $\displaystyle\lim_{x \to a} g(x)$ exist and that *c* is any constant. Then,

1) $\displaystyle\lim_{x \to a}\left[c f(x)\right] = c \lim_{x \to a} f(x)$

In other words we can bring a multiplicative constant out of a limit.

2) $\lim\limits_{x\to a}\left[f(x)\pm g(x)\right]=\lim\limits_{x\to a}f(x)\pm\lim\limits_{x\to a}g(x)$

So to take the limit of a sum or difference all we need to do is take the limit of the individual parts and then put them back together with the appropriate sign. This is also not limited to two functions. This fact will work no matter how many functions we've got separated by "+" or "-".

3) $\lim\limits_{x\to a}\left[f(x)g(x)\right]=\lim\limits_{x\to a}f(x)\ \lim\limits_{x\to a}g(x)$

We take the limits of products in the same way that we can take the limit of sums or differences. Just take the limit of the pieces and then put them back together. Also, as with sums or differences, this fact is not limited to just two functions.

4) $\lim\limits_{x\to a}\left[\dfrac{f(x)}{g(x)}\right]=\dfrac{\lim\limits_{x\to a}f(x)}{\lim\limits_{x\to a}g(x)},\qquad$ provided $\lim\limits_{x\to a}g(x)\neq 0$

As noted in the statement we only need to worry about the limit in the denominator begin zero when we do the limit of a quotient. If it were zero we would end up with a division by zero error and we need to avoid that.

5) $\lim\limits_{x\to a}\left[f(x)\right]^n=\left[\lim\limits_{x\to a}f(x)\right]^n,\qquad$ where n is any real number

In this property n can be any real number (positive, negative, integer, fraction, irrational, zero, *etc.*). In the case that n is an integer this rule can be thought of as a special case of **3)**.

For example consider the case of $n=2$.

$$\begin{aligned}\lim_{x\to a}\left[f(x)\right]^2&=\lim_{x\to a}\left[f(x)\,f(x)\right]\\&=\lim_{x\to a}f(x)\lim_{x\to a}f(x)\qquad\qquad \text{using property 3)}\\&=\left[\lim_{x\to a}f(x)\right]^2\end{aligned}$$

The same can be done for any integer n.

6) $\lim\limits_{x\to a}\left[\sqrt[n]{f(x)}\right]=\sqrt[n]{\lim\limits_{x\to a}f(x)}$

This is just a special case of the previous example.

$$\begin{aligned}\lim_{x\to a}\left[\sqrt[n]{f(x)}\right]&=\lim_{x\to a}\left[f(x)\right]^{\frac{1}{n}}\\&=\left[\lim_{x\to a}f(x)\right]^{\frac{1}{n}}\\&=\sqrt[n]{\lim_{x\to a}f(x)}\end{aligned}$$

7) $\lim_{x \to a} c = c, \qquad c$ is any real number

In other words, the limit of a constant is just the constant. You should be able to convince yourself of this by drawing the graph of *f(x)=c*.

8) $\lim_{x \to a} x = a$

As with the last one you should be able to convince yourself of this by drawing the graph of *f(x)=x*.

9) $\lim_{x \to a} x^n = a^n$

This is really just a special case of property **5)**.

Note that all these properties also hold for the two one-sided limits as well.

Let's compute a limit or two using these properties. The next couple of examples will lead us to some truly useful facts about limits that we will use on a continual basis.

Example 1 Compute the value of the following limit.

$$\lim_{x \to -2} \left(3x^2 + 5x - 9\right)$$

Solution

This first time through we will use only the properties above to compute the limit.

First we will use property **2)** to break up the limit into three separate limits. We will then use property **1)** to bring the constants out of the first two limits. Doing this gives us,

$$\begin{aligned} \lim_{x \to -2} \left(3x^2 + 5x - 9\right) &= \lim_{x \to -2} 3x^2 + \lim_{x \to -2} 5x - \lim_{x \to -2} 9 \\ &= 3\lim_{x \to -2} x^2 + 5\lim_{x \to -2} x - \lim_{x \to -2} 9 \end{aligned}$$

We can now use properties **7)** through **9)** to actually compute the limit.

$$\begin{aligned} \lim_{x \to -2} \left(3x^2 + 5x - 9\right) &= 3\lim_{x \to -2} x^2 + 5\lim_{x \to -2} x - \lim_{x \to -2} 9 \\ &= 3\left(-2\right)^2 + 5\left(-2\right) - 9 \\ &= -7 \end{aligned}$$

Now, let's notice that if we had defined

$$p\left(x\right) = 3x^2 + 5x - 9$$

then the proceeding example would have been,

$$\begin{aligned} \lim_{x \to -2} p\left(x\right) &= \lim_{x \to -2} \left(3x^2 + 5x - 9\right) \\ &= 3\left(-2\right)^2 + 5\left(-2\right) - 9 \\ &= p\left(-2\right) \\ &= -7 \end{aligned}$$

In other words, in this case we were actually able to evaluate the limit by just evaluating the function at the point in question. This seems to violate one of the main concepts about limits that we've seen to this point.

In the previous two sections we made a big deal about the fact that limits do not care about what is happening at the point in question. They only care about what is happening around the point. So how does the previous example fit into this since it appears to violate this main idea about limits?

Despite appearances the limit still doesn't care about what the function is doing at $x=-2$. In this case the function that we've got is simply "nice enough" so that what is happening around the point is exactly the same as what is happening at the point. Eventually we will formalize up just what is meant by "nice enough". At this point let's not worry too much about what "nice enough" is. Let's just take advantage of the fact that some functions will be "nice enough", whatever that means.

The function in the last example was a polynomial. It turns out that all polynomials are "nice enough" so that what is happening around the point is exactly the same as what is happening at the point. This leads to the following fact.

Fact

If $p(x)$ is a polynomial then,

$$\lim_{x\to a} p(x) = p(a)$$

By the end of this section we will generalize this out considerably to most of the functions that we'll be seeing through out this course.

Let's take a look at another example.

Example 2 Evaluate the following limit.

$$\lim_{z\to 1}\frac{6-3z+10z^2}{-2z^4+7z^3+1}$$

Solution

First notice that we can use property **4)** to write the limit as,

$$\lim_{z\to 1}\frac{6-3z+10z^2}{-2z^4+7z^3+1}=\frac{\lim\limits_{z\to 1} 6-3z+10z^2}{\lim\limits_{z\to 1} -2z^4+7z^3+1}$$

Well, actually we should be a little careful. We can do that provided the limit of the denominator isn't zero. As we will see however, it isn't in this case so we're okay.

Now, both the numerator and denominator are polynomials so we can use the fact above to compute the limits of the numerator and the denominator and hence the limit itself.

$$\lim_{z \to 1} \frac{6-3z+10z^2}{-2z^4+7z^3+1} = \frac{6-3(1)+10(1)^2}{-2(1)^4+7(1)^3+1}$$
$$= \frac{13}{6}$$

Notice that the limit of the denominator wasn't zero and so our use of property **4)** was legitimate.

Notice in this last example that again all we really did was evaluate the function at the point in question. So it appears that there is a fairly large class of functions for which this can be done. Let's generalize the fact from above a little.

Fact

Provided *f(x)* is "nice enough" we have,

$$\lim_{x \to a} f(x) = f(a)$$

Again, we will formalize up just what we mean by "nice enough" eventually. At this point all we want to do is worry about which functions are "nice enough". Some functions are "nice enough" for all *x* while others will only be "nice enough" for certain values of *x*. It will all depend on the function.

This fact also holds for the two one-sided limits.

Here is a list of some of the more common functions that are "nice enough".

- Polynomials are nice enough for all *x*'s.
- If $f(x) = \frac{p(x)}{q(x)}$ then *f(x)* will be nice enough provided both *p(x)* and *q(x)* are nice enough and if we are computing $\lim_{x \to a} f(x)$ then $q(a) \neq 0$.
- $\cos(x)$, $\sin(x)$ are nice enough for all *x*'s
- $\sec(x)$, $\tan(x)$ are nice enough provided $x \neq \ldots, -\frac{5\pi}{2}, -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$ In other words secant and tangent are nice enough everywhere cosine isn't zero. To see why recall that these are both really rational functions and that cosine is in the denominator of both then go back up and look at the second bullet above.
- $\csc(x)$, $\cot(x)$ are nice enough provided $x \neq \ldots, -3\pi, -\pi, 0, \pi, 3\pi, \ldots$ In other words cosecant and cotangent are nice enough everywhere sine isn't zero.
- $\sqrt[n]{x}$ is nice enough for all *x* if *n* is odd.
- $\sqrt[n]{x}$ is nice enough for $x \geq 0$ if *n* is even. Here we require $x \geq 0$ to avoid having to deal with complex values.
- a^x, $\mathbf{e}^x$ are nice enough for all *x*'s.

- $\log_b x$, $\ln x$ are nice enough for *x>0*.
- Any sum, difference or product of the above functions will also be nice enough. Quotients will be nice enough provided we don't get division by zero upon evaluating the limit.

The last bullet is important. This means that for any combination of these functions all we need to do is evaluate the function at the point in question, making sure that none of the restrictions are violated. This means that we can now do a large number of limits.

Example 3 Evaluate the following limit.

$$\lim_{x\to 3}\left(-\sqrt[5]{x}+\frac{\mathbf{e}^x}{1+\ln(x)}+\sin(x)\cos(x)\right)$$

Solution

This is a combination of several of the functions listed above and none of the restrictions are violated so all we need to do is plug in *x=3* into the function to get the limit.

$$\begin{aligned}\lim_{x\to 3}\left(-\sqrt[5]{x}+\frac{\mathbf{e}^x}{1+\ln(x)}+\sin(x)\cos(x)\right)&=-\sqrt[5]{3}+\frac{\mathbf{e}^3}{1+\ln(3)}+\sin(3)\cos(3)\\&=8.185427271\end{aligned}$$

Not a very pretty answer, but we can do the limit.

Computing Limits

In the previous section we saw that there is a large class of function that allows us to use

$$\lim_{x\to a} f(x)=f(a)$$

to compute limits. However, there are also many limits for which this won't work easily. The purpose of this section is to develop techniques for dealing with some of these limits that will not allow us to just use this fact.

Let's first got back and take a look at one of the first limits that we looked at and compute its exact value and verify our guess for the limit.

Example 1 Evaluate the following limit.

$$\lim_{x\to 2}\frac{x^2+4x-12}{x^2-2x}$$

Solution

First let's notice that if we try to plug in *x=2* we get,

$$\lim_{x\to 2}\frac{x^2+4x-12}{x^2-2x}=\frac{0}{0}$$

So, we can't just plug in *x=2* to evaluate the limit. So, we're going to have to do something else.

The first thing that we should always do when evaluating limits is to simplify the function as much as possible. In this case that means factoring both the numerator and denominator. Doing this gives,

$$\begin{aligned}\lim_{x\to 2}\frac{x^2+4x-12}{x^2-2x} &= \lim_{x\to 2}\frac{(x-2)(x+6)}{x(x-2)}\\ &= \lim_{x\to 2}\frac{x+6}{x}\end{aligned}$$

So, upon factoring we saw that we could cancel an *x-2* from both the numerator and the denominator. Upon doing this we now have a new rational expression that we can plug *x=2* into. Therefore, the limit is,

$$\lim_{x\to 2}\frac{x^2+4x-12}{x^2-2x} = \lim_{x\to 2}\frac{x+6}{x} = \frac{8}{2} = 4$$

Note that this is in fact what we guessed the limit to be.

On a side note, the 0/0 we got in the previous example is called an **indeterminate form**. This means that we don't really know what it will be until we do some more work. Typically zero in the denominator means it's undefined. However that will only be true if the numerator isn't also zero. Also, zero in the numerator usually means that the fraction is zero, unless the denominator is also zero. Likewise anything divided by itself is 1, unless we're talking about zero.

So, there are really three competing "rules" here and it's not clear which one will win out. It's also possible that none of them will win out and we will get something totally different from undefined, zero, or one. We might, for instance, get a value of 4 out of this to pick a number completely at random.

There are many more kinds of indeterminate forms and we will be discussing indeterminate forms at length in the next chapter.

Let's take a look at a couple of more examples.

Example 2 Evaluate the following limit.

$$\lim_{h\to 0}\frac{2(-3+h)^2-18}{h}$$

Solution

In this case we also get 0/0 and factoring is not really an option. However, there is still some simplification that we can do.

$$\lim_{h\to 0}\frac{2\left(-3+h\right)^2-18}{h}=\lim_{h\to 0}\frac{2\left(9-6h+h^2\right)-18}{h}$$
$$=\lim_{h\to 0}\frac{18-12h+2h^2-18}{h}$$
$$=\lim_{h\to 0}\frac{-12h+2h^2}{h}$$

So, upon multiplying out the first term we get a little cancellation and now notice that we can factor an *h* out of both terms in the numerator which will cancel against the *h* in the denominator and the division by zero problem goes away and we can then evaluate the limit.

$$\lim_{h\to 0}\frac{2\left(-3+h\right)^2-18}{h}=\lim_{h\to 0}\frac{-12h+2h^2}{h}$$
$$=\lim_{h\to 0}\frac{h\left(-12+2h\right)}{h}$$
$$=\lim_{h\to 0}-12+2h$$
$$=-12$$

Example 3 Evaluate the following limit.

$$\lim_{t\to 4}\frac{t-\sqrt{3t+4}}{4-t}$$

Solution

This limit is going to be a little more work than the previous two. Once again however note that we get the indeterminate form 0/0 if we try to just evaluate the limit. Also note that neither of the two examples will be of any help here. We can't factor and we can't just multiply something out to get the function to simplify.

When there is a square root in the numerator or denominator we can try to rationalize and see if that helps. Recall that rationalizing makes use of the fact that

$$\left(a+b\right)\left(a-b\right)=a^2-b^2$$

So, if either the first and/or the second term have a square root in them the rationalizing will eliminate the root(s). This *might* help in evaluating the limit.

Let's try rationalizing the numerator in this case.

$$\lim_{t\to 4}\frac{t-\sqrt{3t+4}}{4-t}=\lim_{t\to 4}\frac{\left(t-\sqrt{3t+4}\right)}{\left(4-t\right)}\frac{\left(t+\sqrt{3t+4}\right)}{\left(t+\sqrt{3t+4}\right)}$$

Remember that to rationalize we just take the numerator (since that's what we're rationalizing), change the sign on the second term and multiply the numerator and denominator by this new term.

Next, we multiply the numerator out being careful to watch minus signs.

$$\lim_{t\to 4}\frac{t-\sqrt{3t+4}}{4-t} = \lim_{t\to 4}\frac{t^2-\left(3t+4\right)}{\left(4-t\right)\left(t+\sqrt{3t+4}\right)}$$

$$= \lim_{t\to 4}\frac{t^2-3t-4}{\left(4-t\right)\left(t+\sqrt{3t+4}\right)}$$

Notice that we didn't multiply the denominator out as well. Most students come out of an Algebra class having it beaten into their heads to always multiply this stuff out. However, in this case multiplying out will make the problem very difficult and in the end you'll just end up factoring it back out.

At this stage we are almost done. Notice that we can factor the numerator so let's do that.

$$\lim_{t\to 4}\frac{t-\sqrt{3t+4}}{4-t} = \lim_{t\to 4}\frac{\left(t-4\right)\left(t+1\right)}{\left(4-t\right)\left(t+\sqrt{3t+4}\right)}$$

At this point all we need to do is notice that if we factor a "-1"out of the first term in the denominator we can do some canceling. At that point the division by zero problem will go away and we can evaluate the limit.

$$\lim_{t\to 4}\frac{t-\sqrt{3t+4}}{4-t} = \lim_{t\to 4}\frac{\left(t-4\right)\left(t+1\right)}{-\left(t-4\right)\left(t+\sqrt{3t+4}\right)}$$

$$= \lim_{t\to 4}\frac{t+1}{-\left(t+\sqrt{3t+4}\right)}$$

$$= -\frac{5}{8}$$

Note that if we had multiplied the denominator out we would not have been able to do this canceling and in all likelihood would not have even seen that some canceling could have been done.

So, we've taken a look at a couple of limits in which evaluation gave the indeterminate form 0/0 and we now have a couple of things to try in these cases.

Let's take a look at another kind of problem that can arise in computing some limits involving piecewise functions.

Example 4 Given the function,

$$g\left(y\right) = \begin{cases} y^2+5 & \text{if } y<-2 \\ 1-3y & \text{if } y\geq -2 \end{cases}$$

Compute the following limits.

(a) $\lim\limits_{y\to 6} g\left(y\right)$

(b) $\lim\limits_{y \to -2} g(y)$

Solution

(a) In this case there really isn't a whole lot to do. In doing limits recall that we must always look at what's happening on both sides of the point in question as we move in towards it. In this case *y=6* is inside the second interval for the function and so there are values of *y* on both sides of *y=6* inside this interval. This means that we can just use the fact to evaluate this limit.

$$\begin{aligned}\lim_{y \to 6} g(y) &= \lim_{y \to 6} 1 - 3y \\ &= -17\end{aligned}$$

(b) This part is the real point to this problem. In this case the point that we want to take the limit for is the cutoff point for the two intervals. In other words we can't just plug *y=-2* into the second portion because this interval does not contain values of *y* to the left of *y=-2* and we need to know what is happening on both sides of the point.

To do this part we are going to have to remember the fact from the section on one-sided limits that says that if the two one-sided limits exist and are the same then the normal limit will also exist and have the same value.

Notice that both of the one sided limits can be done here since we are only going to be looking at one side of the point in question. So let's do the two one-sided limits and see what we get.

$$\begin{aligned}\lim_{y \to -2^-} g(y) &= \lim_{y \to -2^-} y^2 + 5 \qquad \text{since } y \to 2^- \text{ implies } y < -2 \\ &= 9\end{aligned}$$

$$\begin{aligned}\lim_{y \to -2^+} g(y) &= \lim_{y \to -2^+} 1 - 3y \qquad \text{since } y \to 2^+ \text{ implies } y > -2 \\ &= 7\end{aligned}$$

So, in this case we can see that,

$$\lim_{y \to -2^-} g(y) = 9 \neq 7 = \lim_{y \to -2^+} g(y)$$

and so since the two one sided limits aren't the same

$$\lim_{y \to -2} g(y)$$

doesn't exist.

Note that a very simple change to the function will make the limit at *y=-2* exist so don't get in into your head that limits at these cutoff points in piecewise function don't ever exist.

Example 5 Evaluate the following limit.

$$\lim_{y \to -2} g(y) \qquad \text{where, } g(y) = \begin{cases} y^2 + 5 & \text{if } y < -2 \\ 3 - 3y & \text{if } y \geq -2 \end{cases}$$

Solution
The two one-sided limits this time are,

$$\lim_{y\to -2^-} g(y) = \lim_{y\to -2^-} y^2 + 5 \qquad \text{since } y \to 2^- \text{ implies } y < -2$$
$$= 9$$

$$\lim_{y\to -2^+} g(y) = \lim_{y\to -2^-} 3 - 3y \qquad \text{since } y \to 2^+ \text{ implies } y > -2$$
$$= 9$$

The one-sided limits are the same so we get,

$$\lim_{y\to -2} g(y) = 9$$

There is one more limit that we need to do. However, we will need a new fact about limits that will help us to do this.

Fact

If $f(x) \le g(x)$ for all *x* on [*a, b*] and $a \le c \le b$ then,

$$\lim_{x\to c} f(x) \le \lim_{x\to c} g(x)$$

Note that this fact should make some sense to you. If both of the functions are "nice enough" to use the fact then we have,

$$\lim_{x\to c} f(x) = f(c) \le g(c) = \lim_{x\to c} g(x)$$

The inequality is true because we know that *c* is somewhere between *a* and *b* and in that range we also know $f(x) \le g(x)$.

We can take this one step farther to get the following theorem.

Squeeze Theorem

Suppose that for all *x* on [*a, b*] we have,

$$f(x) \le h(x) \le g(x)$$

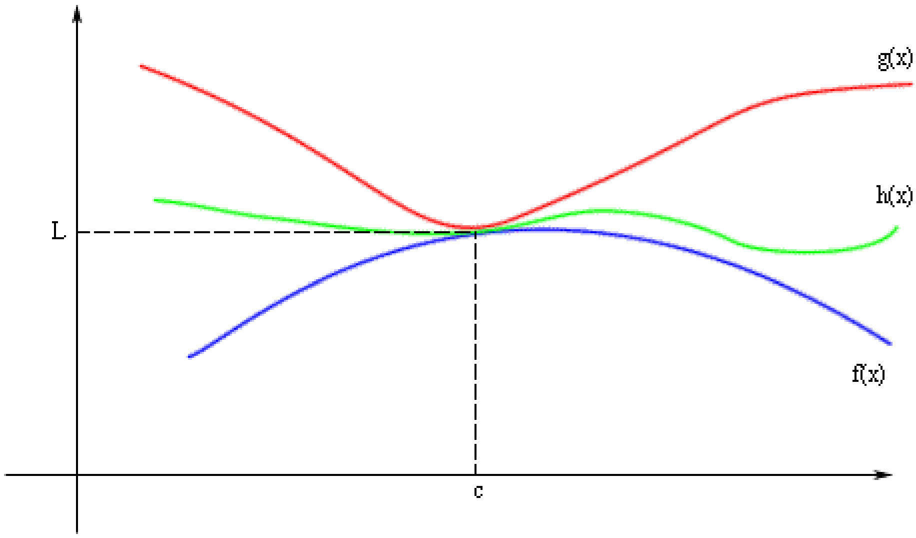
Also suppose that,

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = L$$

for some $a \le c \le b$. Then,

$$\lim_{x\to c} h(x) = L$$

The following figure illustrates what is happening in this theorem.



As we can see from the figure if the limits of *f(x)* and *g(x)* are equal at *x=c* then the function values must also be equal at *x=c*. However, because *h(x)* is "squeezed" between *f(x)* and *g(x)* at this point then *h(x)* must have the same value. Therefore, the limit of *h(x)* at this point must also be the same.

The Squeeze theorem is also known as the **Sandwich Theorem** and the **Pinching Theorem**.

So, how do we use this theorem to help us with limits? Let's take a look at the following example to see the theorem in action.

Example 6 Evaluate the following limit.

$$\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right)$$

Solution

In this example none of the previous examples can help us. There's no factoring or simplifying to do. We can't rationalize and one-sided limits won't work. There's even a question as to whether this limit will exist since we have division by zero inside the cosine at *x=0*.

The first thing to notice is that we know the following about cosine.

$$-1 \leq \cos(x) \leq 1$$

We don't just have an *x* in the cosine, but as long as we avoid *x=0* we can say the same thing for our cosine.

$$-1 \leq \cos\left(\frac{1}{x}\right) \leq 1$$

Its okay for us to ignore *x=0* since we are taking a limit and we know that limits don't care about what's actually going on at *x=0* in this case.

Now if we have the above inequality for our cosine we can just multiply everything by an x^2 and get the following.

$$-x^2 \le x^2 \cos\left(\frac{1}{x}\right) \le x^2$$

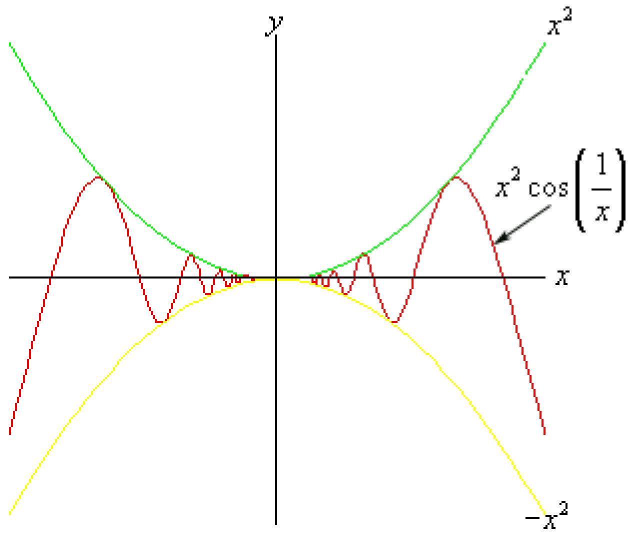
In other words we've managed to squeeze the function that we where interested in between two other function that are very easy to deal with. So, the limits of the two outer functions are.

$$\lim_{x\to 0} x^2 = 0 \qquad\qquad \lim_{x\to 0}\left(-x^2\right) = 0$$

These are the same and so by the Squeeze theorem we must also have,

$$\lim_{x\to 0} x^2\cos\left(\frac{1}{x}\right) = 0$$

We can verify this with the graph of the three functions. This is shown below.



In this section we've seen several tools that we can use to help us to compute limits in which we can't just evaluate the function at the point in question. As we will see many of the limits that we'll be doing in later sections will require one or more of these tools.

Limits Involving Infinity

In this section we will take a look at limits that involve infinity in one way or another. There are two cases that we're going to look at. We'll be looking at limits that equal infinity and limits at infinity.

Limits that equal infinity

We will first look at limits that have infinity as a value. To do this we first need the following definitions.

Definition

We say

$$\lim_{x \to a} f(x) = \infty$$

if we can make *f(x)* arbitrarily large for all *x* sufficiently close to *x=a*, from both sides, without actually letting *x=a*.

We say

$$\lim_{x \to a} f(x) = -\infty$$

if we can make *f(x)* arbitrarily large and negative for all *x* sufficiently close to *x=a*, from both sides, without actually letting *x=a*.

These definitions can be appropriately modified for the one-sided limits as well.

Let's take a look at an example of a limit that gives infinity as a value.

Example 1 Evaluate both of the following limits.

$$\lim_{x \to 3^-} \frac{2x}{x-3} \qquad\qquad \lim_{x \to 3^+} \frac{2x}{x-3}$$

Solution

So we're going to be taking a look at a couple of one-sided limits here. In both cases notice that we can't just plug in *x=3*. If we did we would get division by zero.

Let's take a look at the left-handed limit first. In this case we are going to be assuming that whatever *x* is it will be less than 3. Therefore, as *x* gets closer and closer to *x=3* the numerator is getting closer and closer to 6 while the denominator is getting closer and closer to 0 and will always be negative since we know that whatever *x* is it must satisfy $x<3$.

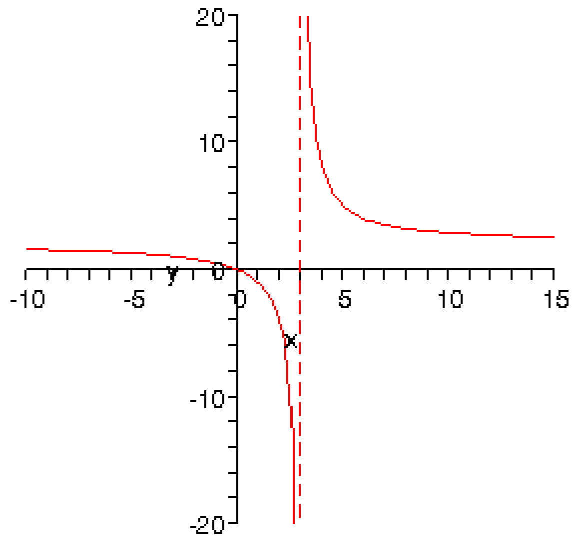
So, as we get closer and closer to *x=3* (from the left) we have a positive, finite number in the numerator divided by an increasingly smaller negative number. This will result in increasing large and negative numbers. In other words,

$$\lim_{x \to 3^-} \frac{2x}{x-3} = -\infty$$

The right-handed limit is similar. As we move in towards *x=3* from the right we will always have $x>3$ and so we will have a positive, finite number in the numerator divided by a increasingly smaller positive number and so the whole thing should be getting larger and larger. In this case the right-handed limit is,

$$\lim_{x \to 3^+} \frac{2x}{x-3} = \infty$$

Below is a graph of this function and it's supports both of our limits.



Note that the limit

$$\lim_{x \to 3} \frac{2x}{x-3}$$

doesn't exist since the two one-sided limits are not the same.

Recall from an Algebra class that we called *x=3* a vertical asymptote. We can define vertical asymptotes in terms of limits.

Definition

The function *f(x)* will have a vertical asymptote at *x=a* if we have any of the following limits at *x=a*.

$$\lim_{x \to a^-} f(x) = \pm\infty \qquad \lim_{x \to a^+} f(x) = \pm\infty \qquad \lim_{x \to a} f(x) = \pm\infty$$

Note that it only requires one of the above limits for a function to have a vertical asymptote at *x=a*.

Limits at infinity

Now that we've seen limits that equal infinity it's time to take a look at limits at infinity. By limits at infinity we mean one of the following two limits.

$$\lim_{x \to \infty} f(x) \qquad \lim_{x \to -\infty} f(x)$$

In other words, we are going to be looking at what happens to a function if we let *x* get very large in either the positive or negative sense. For many of the limits that we're going to be looking at we will need the following fact.

Fact

For *r>0* we have

$$\lim_{x\to\infty}\frac{1}{x^r}=0 \qquad\qquad \lim_{x\to-\infty}\frac{1}{x^r}=0$$

This fact should make sense if you think about it. We require $r>0$ to make sure the term stays in the denominator and as we increase x then x^r will also increase. So, what we end up with is a constant divided by an increasingly large number so the quotient of the two will become increasingly small. In the limit we will get zero.

Let's work an example to see how we do these problems.

Example 2 Evaluate both of the following limits.

$$\lim_{x\to\infty}\frac{2x^4-x^2+8x}{-5x^4+7} \qquad\qquad \lim_{x\to-\infty}\frac{2x^4-x^2+8x}{-5x^4+7}$$

Solution

First, the only difference between these two is that one is going to positive infinity and the other is going to negative infinity.

Our first thought might be to just "plug" in the infinity. However, this will lead to problems. As x goes to infinity a polynomial will behave like the highest power of x behaves (see the work after this example for a quick justification of this). So, the numerator in our case will go to infinity while the denominator will go to minus infinity in the limit. Or,

$$\lim_{x\to\infty}\frac{2x^4-x^2+8x}{-5x^4+7}=\frac{\infty}{-\infty}$$

This is another one of those **indeterminate forms** that we first started seeing in the previous section. We might be tempted to say that the limit is infinity (because of the infinity in the numerator), zero (because of the infinity in the denominator) or -1. However without work there is no way to know which it might be and in fact it might not be any of them, it could be $-\frac{2}{5}$ to pick a number completely at random.

So, when we have a polynomial divided by a polynomial we first identify the largest power of x in the denominator and we then factor this out of both the numerator and denominator. Doing this for the first limit gives,

$$\lim_{x\to\infty}\frac{2x^4-x^2+8x}{-5x^4+7}=\lim_{x\to\infty}\frac{x^4\left(2-\frac{1}{x^2}+\frac{8}{x^3}\right)}{x^4\left(-5+\frac{7}{x^4}\right)}$$

Note that this is really just the same as dividing each term by the largest power of x. Now we can notice that the x^4's will cancel. At this point we can use the fact given above to take the limit of all the terms. This gives,

$$\begin{aligned}\lim_{x\to\infty}\frac{2x^4-x^2+8x}{-5x^4+7} &= \lim_{x\to\infty}\frac{2-\frac{1}{x^2}+\frac{8}{x^3}}{-5+\frac{7}{x^4}}\\ &= \frac{2+0+0}{-5+0}\\ &= -\frac{2}{5}\end{aligned}$$

In this case the indeterminate form was neither of the "obvious" choices of infinity, zero, or -1 so be careful with make these kinds of assumptions with this kind of indeterminate forms.

The second limit is done in a similar fashion. Notice however, that nowhere in the work for the first limit did we actually use the fact that the limit was going to plus infinity. In this case it doesn't matter which infinity we are going towards we will get the same value for the limit.

$$\lim_{x\to-\infty}\frac{2x^4-x^2+8x}{-5x^4+7} = -\frac{2}{5}$$

In the previous example the infinity that we were using in the limit didn't change the answer. This will not always be the case so don't make the assumption that this will always be the case.

In this example we made the statement that a polynomial will always behave as its largest power of *x* will behave as *x* goes to infinity. Let's justify that a little for the numerator. We'll start by factoring out the largest power of *x*.

$$\lim_{x\to\infty} 2x^4-x^2+8x = \lim_{x\to\infty} x^4\left(2-\frac{1}{x^2}+\frac{8}{x^3}\right)$$

Now, we know that the limit of a product is the product of the individual limits so this in turn becomes,

$$\lim_{x\to\infty} 2x^4-x^2+8x = \left(\lim_{x\to\infty} x^4\right)\left(\lim_{x\to\infty}\left(2-\frac{1}{x^2}+\frac{8}{x^3}\right)\right) = \infty\left(2-0+0\right) = \infty$$

As shown we can do each of these limits and we get infinity, which does come from the largest power of *x* in the polynomial. You can always make this kind of argument for any polynomial.

Let's take a look at an example where we get different answers for each limit.

Example 3 Evaluate each of the following limits.

$$\lim_{x\to\infty}\frac{\sqrt{3x^2+6}}{5-2x} \qquad\qquad \lim_{x\to-\infty}\frac{\sqrt{3x^2+6}}{5-2x}$$

Solution

The square root in this problem won't change our work, but it will make the work a little messier.

Let's start with the first limit. In this case the largest power of *x* in the denominator is just an *x*. So we need to factor an *x* out of the numerator and the denominator. When we are done factoring the *x* out we will need an *x* in both of the numerator and the denominator. To get this in the numerator we will have to factor an x^2 out of the square root so that after we take the square root we will get an *x*.

$$\begin{aligned}\lim_{x\to\infty}\frac{\sqrt{3x^2+6}}{5-2x} &= \lim_{x\to\infty}\frac{\sqrt{x^2\left(3+\frac{6}{x^2}\right)}}{x\left(\frac{5}{x}-2\right)}\\ &= \lim_{x\to\infty}\frac{\sqrt{x^2}\sqrt{3+\frac{6}{x^2}}}{x\left(\frac{5}{x}-2\right)}\end{aligned}$$

This is where we need to be careful with the square root in the problem. Don't forget that

$$\sqrt{x^2}=|x|$$

Square roots are ALWAYS positive and so we need the absolute value bars on the *x* to make sure that it will give a positive answer. Using this fact the limit becomes,

$$\lim_{x\to\infty}\frac{\sqrt{3x^2+6}}{5-2x} = \lim_{x\to\infty}\frac{|x|\sqrt{3+\frac{6}{x^2}}}{x\left(\frac{5}{x}-2\right)}$$

Now, we can't just cancel the *x*'s. We first will need to get rid of the absolute value bars. To do this let's recall the definition of absolute value.

$$|x|=\begin{cases} x & \text{if } x\geq 0\\ -x & \text{if } x<0\end{cases}$$

In this case we are going out to plus infinity so we can safely assume that the *x* will be positive and so we can just drop the absolute value bars. The limit is then,

$$\lim_{x\to\infty}\frac{\sqrt{3x^2+6}}{5-2x}=\lim_{x\to\infty}\frac{x\sqrt{3+\frac{6}{x^2}}}{x\left(\frac{5}{x}-2\right)}$$
$$=\lim_{x\to\infty}\frac{\sqrt{3+\frac{6}{x^2}}}{\frac{5}{x}-2}$$
$$=-\frac{\sqrt{3}}{2}$$

Let's now take a look at the second limit. In this case we will need to pay attention to the limit that we are using. The initial work will be the same up until we reach the following step.

$$\lim_{x\to-\infty}\frac{\sqrt{3x^2+6}}{5-2x}=\lim_{x\to-\infty}\frac{|x|\sqrt{3+\frac{6}{x^2}}}{x\left(\frac{5}{x}-2\right)}$$

In this limit we are going to minus infinity so in this case we can assume that x is negative. So, in order to drop the absolute value bars in this case we will need to tack on a minus sign as well. The limit is then,

$$\lim_{x\to-\infty}\frac{\sqrt{3x^2+6}}{5-2x}=\lim_{x\to-\infty}\frac{-x\sqrt{3+\frac{6}{x^2}}}{x\left(\frac{5}{x}-2\right)}$$
$$=\lim_{x\to-\infty}\frac{-\sqrt{3+\frac{6}{x^2}}}{\frac{5}{x}-2}$$
$$=\frac{\sqrt{3}}{2}$$

So, as we saw in the last two examples sometimes the infinity in the limit will affect the answer and other times it won't.

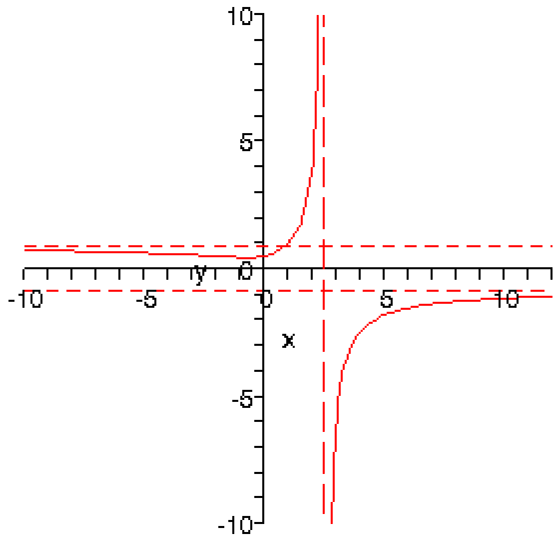
Before moving on to a couple of more examples let's revisit asymptotes. Just as we can have vertical asymptotes defined in terms of limits we can also have horizontal asymptotes defined in terms of limits.

Definition

The function *f(x)* will have a horizontal asymptote at *y*=*L* if either of the following are true.

$$\lim_{x\to\infty} f(x) = L \qquad\qquad \lim_{x\to-\infty} f(x) = L$$

So, the function in the last example will have two horizontal asymptotes. It will also have a vertical asymptote. Here is a graph of the function showing these.



Let's work another example.

Example 4 Evaluate the following limit.

$$\lim_{z\to-\infty} \frac{4z^2+z^6}{1-5z^3}$$

Solution

In this case it looks like we will factor a z^3 out of both the numerator and denominator. Doing this gives,

$$\lim_{z\to-\infty} \frac{4z^2+z^6}{1-5z^3} = \lim_{z\to-\infty} \frac{z^3\left(\frac{4}{z}+z^3\right)}{z^3\left(\frac{1}{z^3}-5\right)}$$

$$= \lim_{z\to-\infty} \frac{\frac{4}{z}+z^3}{\frac{1}{z^3}-5}$$

When we take the limit we'll need to be a little careful. The first term in the numerator and denominator will both be zero. However, the z^3 in the numerator will be going to minus infinity in the limit and so the limit is,

$$\lim_{z\to -\infty} \frac{4z^2 + z^6}{1-5z^3} = \frac{-\infty}{-5} = \infty$$

The final limit is positive because we have a quotient of two negative numbers. However, the division by 5 will not affect the "size" of the answer and so we still get an answer of infinity.

Notice that if we had done the limit as z approaches positive infinity we would have gotten a different answer and so again the limit that we are looking at can have different answers.

We should now take a look at a couple of examples that don't involve polynomials.

Example 5 Evaluate each of the following limits.

$$\lim_{x\to\infty} \mathbf{e}^{x} \qquad \lim_{x\to -\infty} \mathbf{e}^{x} \qquad \lim_{x\to\infty} \mathbf{e}^{-x} \qquad \lim_{x\to -\infty} \mathbf{e}^{-x}$$

Solution

There are really just restatements of facts given in the basic exponential section of the review so I'll leave it to you to go back and verify these.

$$\lim_{x\to\infty} \mathbf{e}^{x} = \infty \qquad \lim_{x\to -\infty} \mathbf{e}^{x} = 0$$

$$\lim_{x\to\infty} \mathbf{e}^{-x} = 0 \qquad \lim_{x\to -\infty} \mathbf{e}^{-x} = \infty$$

Example 6 Evaluate each of the following limits.

$$\lim_{x\to 0^+} \ln x \qquad \lim_{x\to\infty} \ln x$$

Solution

As with the last example I'll leave it to you to verify these restatements from the basic logarithm section.

$$\lim_{x\to 0^+} \ln x = -\infty \qquad \lim_{x\to\infty} \ln x = \infty$$

Note that we had to do a right-handed limit for the first one since we can't plug negative x's into a logarithm. This means that the normal limit won't exist since we must look at x's from both sides of the point in question and x's to the left of zero are negative.

Let's work another example with exponentials that's a little more complicated.

Example 7 Evaluate the following limit.

$$\lim_{x\to\infty} \frac{\mathbf{e}^{4x} - \mathbf{e}^{-2x}}{\mathbf{e}^{7x} + \mathbf{e}^{-x}}$$

Solution

This type of problem works similarly to the quotients involving polynomials. As with those problems if we were to just "plug" in infinity we would get,

$$\lim_{x\to\infty} \frac{\mathbf{e}^{4x} - \mathbf{e}^{-2x}}{\mathbf{e}^{7x} + \mathbf{e}^{-x}} = \frac{\infty}{\infty}$$

So, we will need to do something similar to what we did with the polynomials. In that case we factored out of the numerator and the denominator that term in the denominator that was getting large faster than the other terms (*i.e.* the term with the largest exponent). We'll do the same thing here.

In this case since we are going out to plus infinity we will factor out from both the numerator and denominator the exponential in the denominator with the largest positive exponent. If we were going out to minus infinity we would have factored out the exponential with the largest negative exponent.

$$\begin{aligned}\lim_{x\to\infty}\frac{\mathbf{e}^{4x}-\mathbf{e}^{-2x}}{\mathbf{e}^{7x}+\mathbf{e}^{-x}} &= \lim_{x\to\infty}\frac{\mathbf{e}^{7x}\left(\mathbf{e}^{-3x}-\mathbf{e}^{-9x}\right)}{\mathbf{e}^{7x}\left(1+\mathbf{e}^{-8x}\right)}\\ &= \lim_{x\to\infty}\frac{\mathbf{e}^{-3x}-\mathbf{e}^{-9x}}{1+\mathbf{e}^{-8x}}\end{aligned}$$

Now we use the results from Example 5 above to take the limit of all the pieces. This gives,

$$\lim_{x\to\infty}\frac{\mathbf{e}^{4x}-\mathbf{e}^{-2x}}{\mathbf{e}^{7x}+\mathbf{e}^{-x}} = \lim_{x\to\infty}\frac{0-0}{1+0} = 0$$

There is one final type of example that needs to be worked before we move on.

Example 8 Evaluate the following limit.

$$\lim_{t\to\infty}\left(5t^3-6t^7\right)$$

Solution

Here if we just "plugged" in infinity we would get,

$$\lim_{t\to\infty}\left(5t^3-6t^7\right) = \infty-\infty$$

and as with quotients of infinity it's not clear what this will be. This is yet one more indeterminate form, it could be plus infinity, minus infinity, zero, or some other integer.

To deal with these we factor the smallest power of t from both then take the limit.

$$\begin{aligned}\lim_{t\to\infty}\left(5t^3-6t^7\right) &= \lim_{t\to\infty} t^3\left(5-6t^4\right)\\ &= \left(\infty\right)\left(-\infty\right)\\ &= -\infty\end{aligned}$$

Here the first term in the factored form will go to infinity and the second term will go to minus infinity. This time we can actually get an answer. A very large number times a very large number is still a very large number and so in this case we can do the limit.

We saw quite a few indeterminate forms in this section and we saw how to deal with them. We will run into more indeterminate forms down the road in the next chapter. We will not be able to deal with those in the same way that we did here. Also some of the techniques for dealing with the indeterminate forms in this section will not always work and so we'll need more tools in our arsenal to be able to deal with them.

Continuity

Over the last few sections we've been using the term "nice enough" to define those functions that we could evaluate limits by just evaluating the function at the point in question. It's now time to formally define what we mean by "nice enough".

Definition

A function *f(x)* is said to be **continuous** at $x=a$ if

$$\lim_{x \to a} f(x) = f(a)$$

A function is said to be continuous on the interval $[a, b]$ if it is continuous at each point in the interval.

This definition can be turned around into the following fact.

Fact

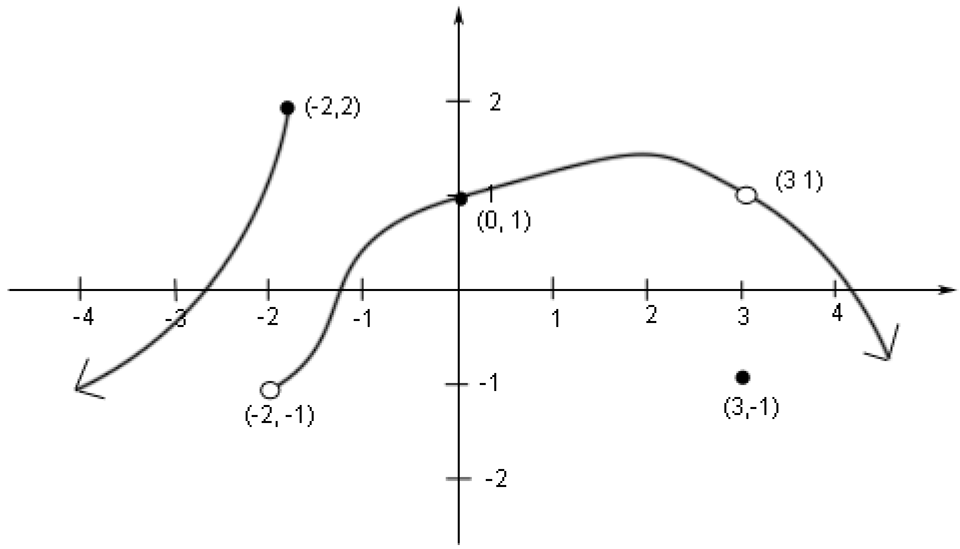
If *f(x)* is continuous at $x=a$ then,

$$\lim_{x \to a} f(x) = f(a)$$

Note that is exactly the same fact that we first put down back when we started looking at limits with the exception that we have replaced the phrase "nice enough" with continuous.

Now, this definition justifies how we've been computing limits for awhile now and so it's good in that sense. However, it doesn't really tell us just what it means for a function to be continuous. Let's take a look at the following example to help us understand just what it means for a function to be continuous.

Example 1 Given the graph of *f(x)* shown below determine if *f(x)* is continuous at $x=-2$, $x=0$, and $x=3$.



Solution
To answer the question for each point we'll need to get both the limit at that point and the function value at that point. If they are equal the function is continuous at that point and if they aren't equal the function isn't continuous at that point.

First x=-2.

$$f(-2) = 2 \qquad\qquad \lim_{x \to -2} f(x) \text{ doesn't exist}$$

The function value and the limit aren't the same and so the function is not continuous at this point. This kind of discontinuity in a graph is called a **jump discontinuity**. Jump discontinuities occur where the graph has a break in it is as this graph does.

Now x=0.

$$f(0) = 1 \qquad\qquad \lim_{x \to 0} f(x) = 1$$

The function is continuous at this point since the function and limit have the same value.

Finally x=3.

$$f(3) = -1 \qquad\qquad \lim_{x \to 3} f(x) = 1$$

The function is not continuous at this point. This kind of discontinuity is called a **removable discontinuity**. Removable discontinuities are those where there is a hole in the graph as there is in this case.

From this example we can get a quick "working" definition of continuity. A function is continuous on an interval if we can draw the graph from start to finish without ever once picking up our pencil. The graph in the last example has only two discontinuities since there are only two places where we would have to pick up our pencil in sketching it.

Example 2 Determine where the function below is not continuous.

$$h(t) = \frac{4t+10}{t^2-2t-15}$$

Solution

Rational functions are continuous everywhere except where we have division by zero. So all that we need to is determine where the denominator is zero.

$$t^2 - 2t - 15 = (t-5)(t+3) = 0$$

The function given will not be continuous at *t=-3* and *t=5*.

A nice consequence of continuity is the following fact.

Fact

If *f(x)* is continuous then,

$$\lim_{x \to a} f\left(g(x)\right) = f\left(\lim_{x \to a} g(x)\right)$$

With this fact we can now do limits like the following example.

Example 3 Evaluate the following limit.

$$\lim_{x \to 0} \mathbf{e}^{\sin x}$$

Solution

Since we know that exponentials are continuous we can use the fact above.

$$\lim_{x \to 0} \mathbf{e}^{\sin x} = \mathbf{e}^{\lim\limits_{x \to 0} \sin x} = \mathbf{e}^{0} = 1$$

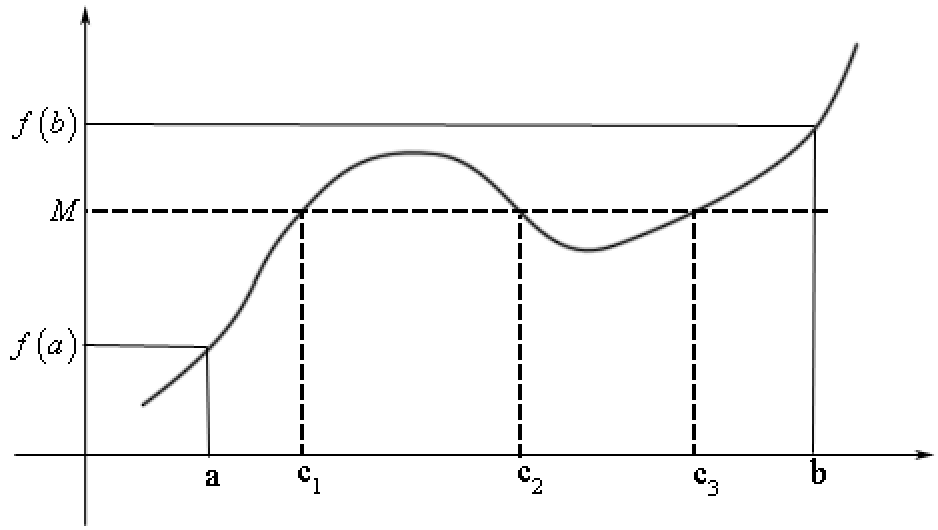
Another very nice consequence of continuity is the Intermediate Value Theorem.

Intermediate Value Theorem

Suppose that *f(x)* is continuous on [*a*, *b*] and let *M* be any number between *f(a)* and *f(b)*. Then there exists a number *c* such that,

1. $a < c < b$
2. $f(c) = M$

All the Intermediate Value Theorem is really saying is that a continuous function will take on all values between *f(a)* and *f(b)*. Below is a graph of a continuous function that illustrates the Intermediate Value Theorem.



As we can see from this image if we pick any value, *M*, that is between the value of *f(a)* and the value of *f(b)* and draw a line straight out from this point the line will hit the graph in at least one point. In other words somewhere between *a* and *b* the function will take on the value of *M*. Note that the figure also shows that it may take on the value more than one place.

It's also important to note that the Intermediate value theorem only says that the function will take on the value of *M* somewhere between *a* and *b*. It doesn't say just what that value will be. It only says that it exists. It also does not tell us how many times the function may take on this value. It only tells us that it takes the value at least once.

A nice use of the Intermediate Value Theorem is to prove the existence of roots of equations.

Example 4 Show that $p(x) = 2x^3 - 5x^2 - 10x + 5$ has a root somewhere in the interval [-1,2].

Solution
What we're really asking here is whether or not the function will take on the value

$$p(x) = 0$$

somewhere between -1 and 2. In other words, we're using *M=0* in the Intermediate value theorem. All we need to show is that 0 is between *p(-1)* and *p(2)* and we'll be done.

$$p(-1) = 8$$
$$p(2) = -19$$

So we have,

$$-19 = p(2) < 0 < p(-1) = 8$$

Therefore 0 is between *p(-1)* and *p(2)* and so by the Intermediate Value Theorem there must be a number *c* somewhere between -1 and 2 so that

$$p(c) = 0$$

Therefore the polynomial does have a root between -1 and 2. For the sake of completeness the root is x=0.4250308563.

The Definition of the Limit

In this section we are going to take a look at the actual definition of the limit.

Definition

Let $f(x)$ be a function defined on an interval that contains $x=a$, except possibly at $x=a$. Then we say that,

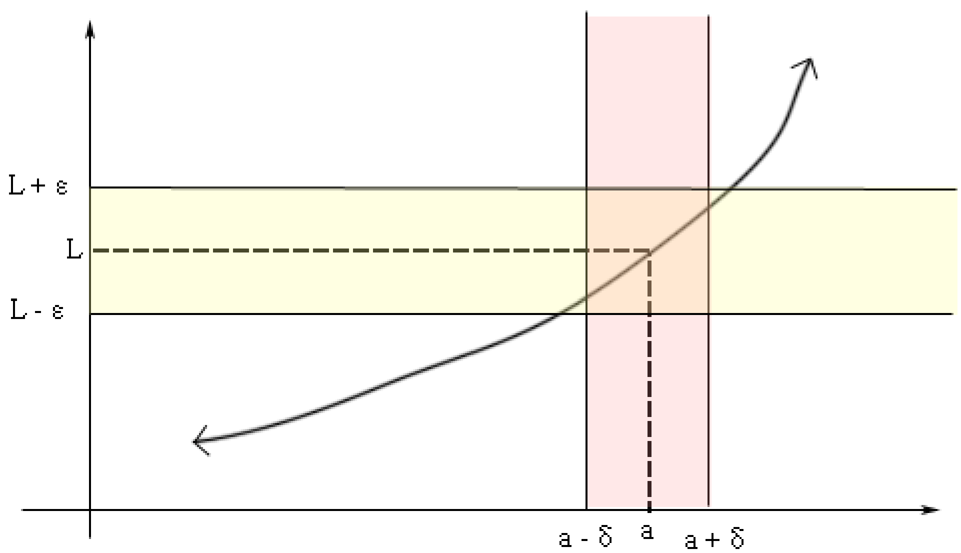
$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is some number $\delta > 0$ such that

$$\left| f(x) - L \right| < \varepsilon \qquad \text{whenever} \qquad \left| x - a \right| < \delta$$

Wow. That's a mouth full. Now that it's written down, just what does this mean?

Let's take a look at the following graph.



Let's assume that the limit does exist. What the definition is telling us is that for **any** number $\varepsilon > 0$ that we pick we can go to our graph and sketch two horizontal lines at $L+\varepsilon$ and $L-\varepsilon$ as shown on the graph above. Then somewhere out there in the world is another number $\delta > 0$ that will allow us to add in two vertical lines to our graph at $a+\delta$ and $a-\delta$.

Now, if we take any x in the pink region this x will be closer to a than either of $a+\delta$ and $a-\delta$. Or,

$$\left|x-a\right|<\delta$$

If we now identify the point on the graph that our choice of x gives then this point on the graph **will** also lie in the intersection of the pink and yellow region. This means that this function value $f(x)$ will be closer to L than either of $L+\varepsilon$ and $L-\varepsilon$. Or,

$$\left|f\left(x\right)-L\right|<\varepsilon$$

So, if we take any value of x in the pink region then the graph for those values of x will lie in the yellow region.

Notice that there are actually an infinite number of possible δ's that we can choose. In fact, if we go back and look at the graph above it looks like we could have taken a slightly larger δ and still gotten the graph from that pink region to be completely contained in the yellow region.

Okay, now that we've gotten the definition out of the way and made an attempt to understand it let's see how it's actually used in practice.

These are a little tricky and it takes a lot of practice to get good at these so don't feel too bad if you don't pick up on this stuff right away. We're going to be looking at two examples that work out fairly easily.

Example 1 Use the definition of the limit to prove the following limit.

$$\lim_{x\to 0} x^2 = 0$$

Solution

In this case both L and a are zero. According to the definition for any number $\varepsilon>0$ we need to find a $\delta>0$ so that the following will be true.

$$\left|x^2-0\right|<\varepsilon \qquad \text{whenever} \qquad \left|x-0\right|<\delta$$

Or upon simplifying things we need,

$$\left|x^2\right|<\varepsilon \qquad \text{whenever} \qquad \left|x\right|<\delta$$

We can bring the exponent out of the absolute value bars in the first inequality.

$$\left|x\right|^2<\varepsilon \qquad \text{whenever} \qquad \left|x\right|<\delta$$

Now, the first inequality will be true provided we have,

$$\left|x\right|<\sqrt{\varepsilon}$$

Therefore, it looks like the definition of the limit requires us to have,

$$|x| < \sqrt{\varepsilon} \qquad \text{whenever} \qquad |x| < \delta$$

In other words, if we choose $\delta = \sqrt{\varepsilon}$ we will have,

$$\left|x^2\right| < \varepsilon \qquad \text{whenever} \qquad |x| < \sqrt{\varepsilon}$$

So, we've managed to find a δ that will work for any ε that we choose and so we've managed to prove the limit using the definition.

Let's take a look at a slightly more complicated limit.

Example 2 Use the definition of the limit to prove the following limit.

$$\lim_{x \to 2} 10x - 6 = 14$$

Solution

We'll start this one out the same way that we did the first one. The definition of the limit requires that for any number $\varepsilon > 0$ we need to find a $\delta > 0$ so that the following will be true.

$$\left|(10x - 6) - 14\right| < \varepsilon \qquad \text{whenever} \qquad |x - 2| < \delta$$

The first inequality can be simplified a little however.

$$|10x - 20| < \varepsilon$$
$$10|x - 2| < \varepsilon$$

So, the definition of the limit requires that,

$$|x - 2| < \frac{\varepsilon}{10} \qquad \text{whenever} \qquad |x - 2| < \delta$$

Therefore, if we choose $\delta = \frac{\varepsilon}{10}$ we will get,

$$\left|(10x - 6) - 14\right| < \varepsilon \qquad \text{whenever} \qquad |x - 2| < \frac{\varepsilon}{10}$$

So, as in the previous example we've managed to find a δ that will work for any ε that we choose and so we've managed to prove the limit using the definition.

Using the definition of the limit to prove limits is a difficult process to understand the first couple of times around. Do not feel bad if you don't get this stuff right away. It's very common to not understand this right away.

To make matters worse, the ones that we worked here where actually fairly simplistic. Most limits are much more difficult to prove using the definition.

Derivatives

Introduction

In this chapter we will start looking at the next major topic in a calculus class. We will be looking at derivatives in this chapter (as well as the next chapter). This chapter is devoted almost exclusively to finding derivatives. We will be looking at one application of them in this chapter. We will be leaving most of the applications of derivatives to the next chapter.

Here is a listing of the topics covered in this chapter.

The Definition of the Derivative – In this section we will be looking at the definition of the derivative.

Interpretation of the Derivative – Here we will take a quick look at some interpretations of the derivative.

Differentiation Formulas – Here we will start introducing some of the differentiation formulas used in a calculus course.

Product and Quotient Rule – In this section we will took at differentiating products and quotients of functions.

Derivatives of Trig Functions – We'll give the derivatives of the trig functions in this section.

Derivatives of Exponential and Logarithm Functions – In this section we will get the derivatives of the exponential and logarithm function.

Derivatives of Inverse Trig Functions – Here we will look at the derivatives of inverse trig functions.

Derivatives of Hyperbolic Trig Functions – Here we will look at the derivatives of hyperbolic trig functions.

Chain Rule – The Chain Rule is one of the more important differentiation rules and will allow us to differentiate a wider variety of functions. In this section we will take a look at it.

Implicit Differentiation – In this section we will be looking at implicit differentiation. Without this we won't be able to work some of the applications of derivatives.

Related Rates – In this section we will look at the lone application to derivatives in this chapter. This topic is here rather than the next chapter because it will help to cement in our minds on of the more important concepts about derivatives and because it requires implicit differentiation.

Higher Order Derivatives – Here we will introduce the idea of higher order derivatives.

Logarithmic Differentiation – The topic of logarithmic differentiation is not always presented in a standard calculus course. It is presented here for those how are interested in seeing how it is done and the types of functions on which it can be used.

The Definition of the Derivative

In the first section of the last chapter we saw that the computation of the slope of a tangent line, the instantaneous rate of change of a function, and the instantaneous velocity of an object at $x=a$ all required us to compute the following limit.

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

We also saw that with a small change of notation this limit could also be written as,

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \qquad \textbf{(1)}$$

This is such an important limit and it arises in so many places that we give it a name. We call it a **derivative**. Here is the official definition of the derivative.

Definition

The **derivative of** $f(x)$ **with respect to *x*** is the function $f'(x)$ and is defined as,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad \textbf{(2)}$$

Note that we replaced all the a's in (1) with x's to acknowledge the fact that the derivative is really a function as well. We often "read" $f'(x)$ as "f prime of x".

Let's compute a couple of derivatives using the definition.

Example 1 Find the derivative of the following function using the definition of the derivative.

$$f(x) = 2x^2 - 16x + 35$$

Solution

So, all we really need to do is to plug this function into the definition of the derivative and do some algebra. Admittedly, the algebra will get somewhat unpleasant at times, but it's just algebra.

So, first plug the function into the definition of the derivative.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0} \frac{2(x+h)^2 - 16(x+h) + 35 - (2x^2 - 16x + 35)}{h}$$

Be careful and make sure that you properly deal with parenthesis when doing the subtracting.

Now, we know from the previous chapter that we can't just plug in *h=0* since this will give us a division by zero error. So we are going to have to do some work. In this case that means multiplying everything out and distributing the minus sign through on the second term. Doing this gives,

$$f'(x) = \lim_{h\to 0} \frac{2x^2 + 4xh + 2h^2 - 16x - 16h + 35 - 2x^2 + 16x - 35}{h}$$
$$= \lim_{h\to 0} \frac{4xh + 2h^2 - 16h}{h}$$

Notice that every term in the numerator that didn't have an h in it canceled out and we can now factor an h out of the numerator which will cancel against the h in the denominator. After that we can compute the limit.

$$f'(x) = \lim_{h\to 0} \frac{h(4x + 2h - 16)}{h}$$
$$= \lim_{h\to 0} 4x + 2h - 16$$
$$= 4x - 16$$

So, the derivative is,

$$f'(x) = 4x - 16$$

Example 2 Find the derivative of the following function using the definition of the derivative.

$$g(t) = \frac{t}{t+1}$$

Solution

This one is going to be a little messier as far as the algebra goes. However, outside of that it will work in exactly the same manner as the previous examples.

$$g'(t) = \lim_{h\to 0} \frac{g(t+h) - g(t)}{h}$$
$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{t+h}{t+h+1} - \frac{t}{t+1}\right)$$

First note that we changed all the letters in the definition to match up with the given function. Next, also note that we wrote the fraction a much more compact manner to help us with the work.

As with the first problem we can't just plug in $h=0$. So we will need to simplify things a little. In this case we will need to combine the two terms in the numerator into a single rational expression.

$$\begin{aligned} g'(t) &= \lim_{h\to 0}\frac{1}{h}\left(\frac{(t+h)(t+1)-t(t+h+1)}{(t+h+1)(t+1)}\right) \\ &= \lim_{h\to 0}\frac{1}{h}\left(\frac{t^2+t+th+h-(t^2+th+t)}{(t+h+1)(t+1)}\right) \\ &= \lim_{h\to 0}\frac{1}{h}\left(\frac{h}{(t+h+1)(t+1)}\right) \end{aligned}$$

Before finishing this let's note a couple of things. First, we didn't multiply out the denominator. Multiplying out the denominator will just overly complicate things so let's keep it simple. Next, as with the first example, after the simplification we only have terms with h's in them left in the numerator.

So, upon canceling the h we can evaluate the limit and get the derivative.

$$\begin{aligned} g'(t) &= \lim_{h\to 0}\frac{1}{(t+h+1)(t+1)} \\ &= \frac{1}{(t+1)(t+1)} \\ &= \frac{1}{(t+1)^2} \end{aligned}$$

The derivative is

$$g'(t) = \frac{1}{(t+1)^2}$$

Example 3 Find the derivative of the following function using the derivative.

$$R(z) = \sqrt{5z-8}$$

Solution

First plug into the definition of the derivative as we've done with the previous two examples.

$$\begin{aligned} R'(z) &= \lim_{h\to 0}\frac{R(z+h)-R(z)}{h} \\ &= \lim_{h\to 0}\frac{\sqrt{5(z+h)-8}-\sqrt{5z-8}}{h} \end{aligned}$$

In this problem we're going to have to rationalize the numerator.

$$R'(z) = \lim_{h \to 0} \frac{\left(\sqrt{5(z+h)-8} - \sqrt{5z-8}\right)}{h} \frac{\left(\sqrt{5(z+h)-8} + \sqrt{5z-8}\right)}{\left(\sqrt{5(z+h)-8} + \sqrt{5z-8}\right)}$$

$$= \lim_{h \to 0} \frac{5z+5h-8-(5z-8)}{h\left(\sqrt{5(z+h)-8} + \sqrt{5z-8}\right)}$$

$$= \lim_{h \to 0} \frac{5h}{h\left(\sqrt{5(z+h)-8} + \sqrt{5z-8}\right)}$$

Again, after the simplification we have only *h*'s left in the numerator. So, cancel the *h* and evaluate the limit.

$$R'(z) = \lim_{h \to 0} \frac{5}{\sqrt{5(z+h)-8} + \sqrt{5z-8}}$$

$$= \frac{5}{\sqrt{5z-8} + \sqrt{5z-8}}$$

$$= \frac{5}{2\sqrt{5z-8}}$$

And so we get a derivative of,

$$R'(z) = \frac{5}{2\sqrt{5z-8}}$$

Let's work one more example. This one will be a little different, but it's got a point that needs to be made.

Example 4 Determine $f'(0)$ for $f(x) = |x|$

Solution

Since this problem is asking for the derivative at a specific point we'll go ahead and use that in our work. It will make our life easier and that's always a good thing.

So, plug into the definition and simplify.

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{|0+h| - |0|}{h}$$

$$= \lim_{h \to 0} \frac{|h|}{h}$$

We saw a situation like this back when we were looking at limits involving infinity. As in that section we can't just cancel the h's. We will have to look at the two one sided limits and recall that

$$\left| h \right| = \begin{cases} h & \text{if } h \geq 0 \\ -h & \text{if } h < 0 \end{cases}$$

$$\begin{aligned} \lim_{h \to 0^-} \frac{\left| h \right|}{h} &= \lim_{h \to 0^-} \frac{-h}{h} && \text{because } h < 0 \text{ in a left-hand limit.} \\ &= \lim_{h \to 0^-} \left(-1 \right) \\ &= -1 \\ \lim_{h \to 0^+} \frac{\left| h \right|}{h} &= \lim_{h \to 0^+} \frac{h}{h} && \text{because } h > 0 \text{ in a left-hand limit.} \\ &= \lim_{h \to 0^+} 1 \\ &= 1 \end{aligned}$$

The two one-sided limits are different and so

$$\lim_{h \to 0} \frac{\left| h \right|}{h}$$

doesn't exist. However, this is the limit that gives us the derivative that we're after.

If the limit doesn't exist then the derivative doesn't exist either.

In this example we have finally seen a function for which the derivative doesn't exist at a point. This is a fact of life that we've got to be aware of. Derivatives will not always exist. Note as well that this doesn't say anything about whether or not the derivative exists anywhere else. In fact, for the absolute value function the derivative exists at every point except $x=0$.

The preceding discussion leads to the following definition.

Definition

A function $f(x)$ is called **differentiable** at $x=a$ if $f'(a)$ exists and $f(x)$ is called differentiable on an interval if the derivative exists for each point in that interval.

Alternate Notation

Next we need to discuss some alternate notation for the derivative. The typical derivative notation is the "prime" notation. However, there is another notation that is used on occasion so let's cover that.

Given a function $y=f(x)$ all of the following are equivalent and represent the derivative of $f(x)$ with respect to x.

$$f'(x) = y' = \frac{df}{dx} = \frac{dy}{dx} = \frac{d}{dx}\left(f(x) \right) = \frac{d}{dx}\left(y \right)$$

Because we also need to evaluate derivatives on occasion we also need a notation for evaluating derivatives when using the fractional notation. So if we want to evaluate the derivative at $x=a$ all of the following are equivalent.

$$f'(a) = y'\big|_{x=a} = \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{dy}{dx}\right|_{x=a}$$

Note as well that on occasion we will drop the *(x)* part on the function to simplify the notation somewhat. In these cases the following are equivalent.

$$f'(x) = f'$$

Interpretations of the Derivative

Before moving on to the section where we learn how we actually take derivatives we need to take a quick look at some of the interpretations of the derivative. All of these interpretations arise from recalling how our definition of the derivative came about. The definition came about by noticing that all the problems that we worked in the first section in the chapter on limits required us to evaluate the same limit.

Rate of Change

The first interpretation of a derivative is rate of change. This was not the first problem that we looked at involving the limit, but it is the most important interpretation of the derivative. If $f(x)$ represents a quantity at any x then the derivative $f'(x)$ represents the instantaneous rate of change of $f(a)$ at $x=a$.

Example 1 Suppose that the amount of water in a holding tank at t minutes is given by $V(t) = 2t^2 - 16t + 35$. Determine each of the following.

(a) Is the volume of water in the tank increasing or decreasing at $t=1$ minute?
(b) Is the volume of water in the tank increasing or decreasing at $t=5$ minutes?
(c) Is the volume of water in the tank changing faster at $t=1$ or $t=5$ minutes?
(d) Is the volume of water in the tank ever not changing? If so, when?

Solution

In the solution to this example we will use both notations for the derivative just to get you familiar with the different notations.

First, notice that the function giving the volume of water in the tank is the same function that we saw in Example 1 in the last section except the letters have changed.

We are going to need the rate of change of the volume to answer these questions. This means that we will need the derivative of this function since that will give us a formula for the rate of change at any time t. The change in letters between the function in this example versus the function in the example from the last section won't affect the work and so we can just use the answer from that example with an appropriate change in letters.

The derivative is.

$$V'(t) = 4t - 16 \qquad \text{OR} \qquad \frac{dV}{dt} = 4t - 16$$

Recall from our work in the first limits section that we determined that if the rate of change was positive then the quantity was increasing and if the rate of change was negative then the quantity was decreasing.

We can now work the problem.
(a) In this case all that we need is the rate of change of the volume at *t=1* or,

$$V'(1) = -12 \qquad \text{OR} \qquad \left.\frac{dV}{dt}\right|_{t=1} = -12$$

So, at *t=1* the rate of change is negative and so the volume must be decreasing at this time.

(b) Again, we will need the rate of change at *t=5*.

$$V'(5) = 4 \qquad \text{OR} \qquad \left.\frac{dV}{dt}\right|_{t=5} = 4$$

In this case the rate of change is positive and so the volume must be increasing at *t=5*.

(c) To answer this question all that we look at is the size of the rate of change and we don't worry about the sign of the rate of change. All that we need to know here is that the larger the number the faster the rate of change. So, in this case the volume is changing faster at *t=1* than at *t=5*.

(d) The volume will not be changing if it has a rate of change of zero. To answer this question we will then need to solve

$$V'(t) = 0 \qquad \text{OR} \qquad \frac{dV}{dt} = 0$$

This is easy enough to do.

$$4t - 16 = 0 \qquad \Rightarrow \qquad t = 4$$

So at *t=4* the volume isn't changing. Note that all this is saying is that for a brief instant the volume isn't changing. It doesn't say that at this point the volume will quit changing permanently.

If we go back to our answers from parts (a) and (b) we can get an idea about what is going on. At *t=1* the volume is decreasing and at *t=5* the volume is increasing. So at some point in time the volume needs to switch from decreasing to increasing. That time is *t=4*.

This is the time in which the volume goes from decreasing to increasing and so for the briefest instant in time the volume will quit changing as it changes from decreasing to increasing.

Note that one of the more common mistakes that students make in these kinds of problems is to try and determine increasing/decreasing from the function values rather than the derivatives. In this case if we took the function values at $t=1$ and $t=5$ we would get,

$$V(1) = 21 \qquad\qquad V(5) = 5$$

Clearly the volume has decreased from $t=1$ to $t=5$, but that doesn't say anything about whether or not the function is increasing or decreasing at the two points in question. We aren't looking for whether or not the overall volume has decreased or increased. We are looking for whether or not the volume is decreasing or increases exactly at the particular times in question.

So, be careful. When asked to determine increasing or decreasing make sure and look at the derivative. It is the only sure way to get the correct answer.

Slope of Tangent Line

This is the next major interpretation of the derivative. The slope of the tangent line to $f(x)$ at $x=a$ is $f'(a)$. The tangent line is given by,

$$y = f(a) + f'(a)(x-a)$$

Example 2 Find the tangent line to the following function at $z=3$.

$$R(z) = \sqrt{5z-8}$$

Solution

We first need the derivative of the function and we found that in Example 3 in the last section. The derivative is,

$$R'(z) = \frac{5}{2\sqrt{5z-8}}$$

Now all that we need is the function value and derivative (for the slope) at $z=3$.

$$R(3) = \sqrt{7} \qquad\qquad m = R'(3) = \frac{5}{2\sqrt{7}}$$

The tangent line is then,

$$y = \sqrt{7} + \frac{5}{2\sqrt{7}}(z-3)$$

Velocity

Recall that this can be thought of as a special case of the rate of change interpretation. If the position of an object is given by *f(t)* after *t* units of time the velocity of the object at *t=a* is given by *f'(a)*.

Example 3 Suppose that the position of an object after *t* hours is given by,

$$g(t) = \frac{t}{t+1}$$

Answer both of the following about this object.

(a) Is the object moving to the right or the left at *t=10* hours?

(b) Does the object ever stop moving?

Solution

Once again we need the derivative and we found that in Example 2 in the last section. The derivative is,

$$g'(t) = \frac{1}{(t+1)^2}$$

(a) To determine if the object is moving to the right (velocity is positive) or left (velocity is negative) we need the derivative at *t=10*.

$$g'(10) = \frac{1}{121}$$

So the velocity at *t=10* is positive and so the object is moving to the right at *t=10*.

(b) The object will stop moving if the velocity is ever zero. However, note that the only way a rational expression will ever be zero is if the numerator is zero. Since the numerator of the derivative (and hence the speed) is a constant it can't be zero.

Therefore, the velocity will never stop moving.

In fact, we can say a little more here. The object will always be moving to the right since the velocity is always positive.

We've seen three major interpretations of the derivative here. You will need to remember these, especially the rate of change, as they will show up continually through out this course.

Differentiation Formulas

In the first section of this chapter we saw the definition of the derivative and we computed a couple of derivatives using the definition. As we saw in those examples there was a fair amount of work involved in computing the limits and the functions that we worked with were not terribly complicated.

For more complex functions using the definition of the derivative would be an almost impossible task. Luckily for us we won't have to use the definition terribly often. We will have to use it on occasion, however we have a large collection of formulas and properties that we can use to simplify our life considerably and will allow us to avoid using the definition whenever possible.

We will introduce most of these formulas over the course of the next several sections. We will start in this section with some of the basic properties and formulas. We will give the properties and formulas in this section in both "prime" notation and "fraction" notation.

Properties

1) $\left(f(x) \pm g(x) \right)' = f'(x) \pm g'(x)$ **OR** $\dfrac{d}{dx}\left(f(x) \pm g(x) \right) = \dfrac{df}{dx} \pm \dfrac{dg}{dx}$

In other words, to differentiate a sum or difference all we need to do is differentiate the individual terms and then put them back together with the appropriate signs. Note as well that this property is not limited to two functions.

This property is easy enough to show using the definition of the derivative. We'll do this for the sum of two functions and we'll leave it to you to check the difference of two functions.

We first plug the sum into the definition of the derivative and rewrite the numerator a little.

$$\begin{aligned} \left(f(x) + g(x) \right)' &= \lim_{h \to 0} \frac{f(x+h) + g(x+h) - \left(f(x) + g(x) \right)}{h} \\ &= \lim_{h \to 0} \frac{f(x+h) - f(x) + g(x+h) - g(x)}{h} \end{aligned}$$

Now, break up the fraction into two pieces and recall that the limit of a sum is the sum of the limits. Using this fact we see that we end up with the definition of the derivative for each of the two functions.

$$\begin{aligned} \left(f(x) + g(x) \right)' &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\ &= f'(x) + g'(x) \end{aligned}$$

2) $\left(cf(x) \right)' = cf'(x)$ **OR** $\dfrac{d}{dx}\left(cf(x) \right) = c\dfrac{df}{dx}$, c is any number

So, we can factor a multiplicative constant out of a derivative if we need to.

This is also very easy to show using the definition provided you recall that we can factor a constant out of a limit.

$$\begin{aligned}\left(cf(x)\right)' &= \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h}\\ &= c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= cf'(x)\end{aligned}$$

Note that we have not included formulas for the derivative of products or quotients of two functions in these properties. The derivative of a product or quotient of two functions is not the product or quotient of the derivatives of the individual pieces. We will take a look at these in the next section.

Facts

1) If $f(x)=c$ then $f'(x)=0$ **OR** $\frac{d}{dx}(c)=0$

The derivative of a constant is zero.

This can be easily checked using the definition of the derivative.

$$\begin{aligned}\left(f(x)\right)' &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{c-c}{h}\\ &= \lim_{h\to 0}0\\ &= 0\end{aligned}$$

2) If $f(x)=x^n$ then $f'(x)=nx^{n-1}$ **OR** $\frac{d}{dx}(x^n)=nx^{n-1}$, n is any number.

This formula is sometimes called the **power rule**. All we are doing here is bringing the original exponent down in front and multiplying and then subtracting one from the original exponent.

Note as well that in order to use this formula n must be a number, it can't be a variable and the base must be a variable.

Showing this using the definition of the derivative is not terribly difficult, but is a little tedious and so we won't show it here.

These are the only properties and formulas that we'll give in this section. Let's do compute some derivatives using these properties.

Example 1 Differentiate each of the following functions.

(a) $f(x)=15x^{100}-3x^{12}+5x-46$

(b) $g(t)=2t^6+7t^{-6}$

(c) $y = 8z^3 - \dfrac{1}{3z^5} + z - 23$

(d) $T(x) = \sqrt{x} + 9\sqrt[3]{x^7} - \dfrac{2}{\sqrt[5]{x^2}}$

(e) $h(x) = x^{\pi} - x^{\sqrt{2}}$

Solution

(a) In this case we have the sum and difference of four terms and so we will differentiate each of the terms using the two facts and then put them back together with the proper sign. So, here is the derivative for this function.

$$\begin{aligned} f'(x) &= 15(100)x^{99} - 3(12)x^{11} + 5(1)x^0 - 0 \\ &= 1500x^{99} - 36x^{11} + 5 \end{aligned}$$

Notice that in the third term the exponent was a one and so upon subtracting 1 from the original exponent we get a new exponent of zero. Now recall that $x^0=1$. Also notice that in each term where there was a coefficient we just multiplied the coefficient times the derivative of the x term.

(b) The point of this problem is to make sure that you deal with negative exponents correctly. Here is the derivative.

$$\begin{aligned} g'(t) &= 2(6)t^5 + 7(-6)t^{-7} \\ &= 12t^5 - 42t^{-7} \end{aligned}$$

Make sure that you correctly deal with the exponents in these cases, especially the negative exponents.

(c) Now in this function the second term is not correctly set up for us to use the power rule. The power rule requires that the term be a variable to a power only and the term must be in the numerator.

So, prior to differentiating we first need to rewrite the second term into a form that we can deal with.

$$y = 8z^3 - \frac{1}{3}z^{-5} + z - 23$$

Note that we left the 3 in the denominator and only moved the variable up to the numerator. This is usually what you will want to do in these cases. We can now differentiate the function.

$$y' = 24z^2 + \frac{5}{3}z^{-6} + 1$$

(d) All of the terms in this function have roots in them. In order to use the power rule we need to first convert all the roots to fractional exponents.

$$\begin{aligned} T(x) &= x^{\frac{1}{2}} + 9\left(x^7\right)^{\frac{1}{3}} - \frac{2}{\left(x^2\right)^{\frac{1}{5}}} \\ &= x^{\frac{1}{2}} + 9x^{\frac{7}{3}} - \frac{2}{x^{\frac{2}{5}}} \\ &= x^{\frac{1}{2}} + 9x^{\frac{7}{3}} - 2x^{-\frac{2}{5}} \end{aligned}$$

In the last two terms we combined the exponents. You should always do this. Also we moved the term in the denominator of the third term up to the numerator. We can now differentiate the function.

$$\begin{aligned} T'(x) &= \frac{1}{2}x^{-\frac{1}{2}} + 9\left(\frac{7}{3}\right)x^{\frac{4}{3}} - 2\left(-\frac{2}{5}\right)x^{-\frac{7}{5}} \\ &= \frac{1}{2}x^{-\frac{1}{2}} + \frac{63}{3}x^{\frac{4}{3}} + \frac{4}{5}x^{-\frac{7}{5}} \end{aligned}$$

Make sure that you can deal with fractions. You will see a lot of them in this class.

(e) In all of the previous examples the exponents have been nice integers or fractions. That is usually what we'll see in this class. However, the exponent only needs to be a number so don't get excited about problems like this one. They work exactly the same.

$$h'(x) = \pi x^{\pi-1} - \sqrt{2}x^{\sqrt{2}-1}$$

The answer is a little messy and we won't reduce the exponents down to decimals. However, this problem is not terribly difficult it just looks that way initially.

There is a general rule about derivatives in this class that you will need to get into the habit of using. When you see radicals you should always first convert the radical to a fractional exponent and then simplify exponents as much as possible. Following this rule will save you a lot of grief in the future.

Back when we first put down the properties we noted that we hadn't included a property for products and quotients. That doesn't mean that we can't differentiate any product or quotient at this point. There are some that we can do.

Example 2 Differentiate each of the following functions.

(a) $y = \sqrt[3]{x^2}\left(2x - x^2\right)$

(b) $h(t) = \dfrac{2t^5 + t^2 - 5}{t^2}$

Solution

(a) In this function we can't just differentiate the first term, differentiate the second term and then multiply the two back together. That just won't work. We will discuss this in detail in the next section so if you're not sure you believe that hold on for a bit and we'll be looking at that soon.

It is still possible to do this derivative however. All that we need to do is convert the radical to fractional exponents (as we should anyway) and then multiply this through the parenthesis.

$$y = x^{\frac{2}{3}}\left(2x - x^2\right) = 2x^{\frac{5}{3}} - x^{\frac{8}{3}}$$

Now we can differentiate the function.

$$y' = \frac{10}{3}x^{\frac{2}{3}} - \frac{8}{3}x^{\frac{5}{3}}$$

(b) As with the first part we can't just differentiate the numerator and the denominator and the put it back together as a fraction. Again, if you're not sure you believe this hold on until the next section and we'll take a more detailed look at this.

We can simplify this rational expression however as follows.

$$h(t) = 2t^3 + 1 - \frac{5}{t^2} = 2t^3 + 1 - 5t^{-2}$$

This is a function that we can differentiate.

$$h'(t) = 6t^2 + 10t^{-3}$$

So, as we saw in this example there are a few products and quotients that we can differentiate. If we can first do some simplification the functions will sometimes simplify into a form that can be differentiated using the properties and formulas in this section.

Before moving on to the next section let's work a couple of examples to remind us once again of some of the interpretations of the derivative.

Example 3 Find the equation of the tangent line to $f(x) = 4x - 8\sqrt{x}$ at *x=16*.

Solution

We know that the equation of a tangent line is given by,

$$y = f(a) + f'(a)(x - a)$$

So, we will need the derivative of the function.

$$f(x) = 4x - 8x^{\frac{1}{2}}$$

$$f'(x) = 4 - 4x^{-\frac{1}{2}} = 4 - \frac{4}{x^{\frac{1}{2}}}$$

Now we need to evaluate the function and the derivative.

$$f(16) = 64 - 8(4) = 32$$

$$f'(x) = 4 - \frac{4}{4} = 3$$

The tangent line is then,

$$y = 32 + 3(x - 16) = 3x - 16$$

Example 4 The position of an object at any time t (in hours) is given by,

$$s(t) = 2t^3 - 21t^2 + 60t - 10$$

Determine when the object is moving to the right and when the object is moving to the left.

Solution

The only way that we'll know for sure which direction the object is moving is to have the velocity in hand. Recall that if the velocity is positive the object is moving off to the right and if the velocity is negative then the object is moving to the left.

So, we need the derivative since the derivative is the velocity of the object. The derivative is,

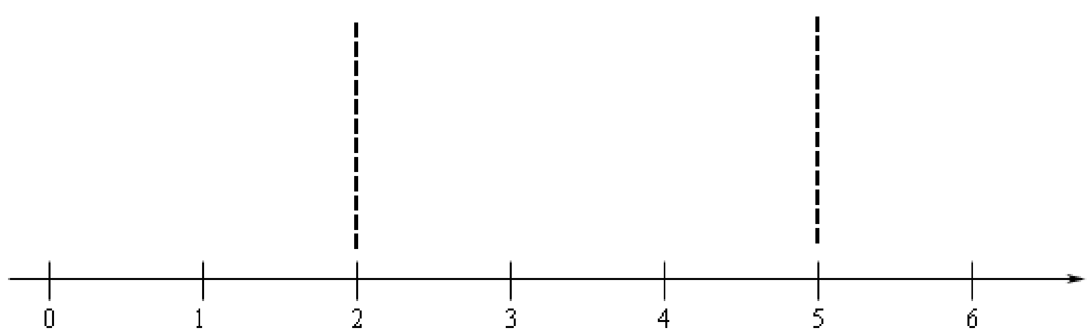
$$s'(t) = 6t^2 - 42t + 60 = 6(t^2 - 7t + 10) = 6(t-2)(t-5)$$

The reason for factoring the derivative will be apparent shortly.

Now, we need to determine where the derivative is positive and where the derivative is negative. There are several ways to do this. The method that I tend to prefer is the following.

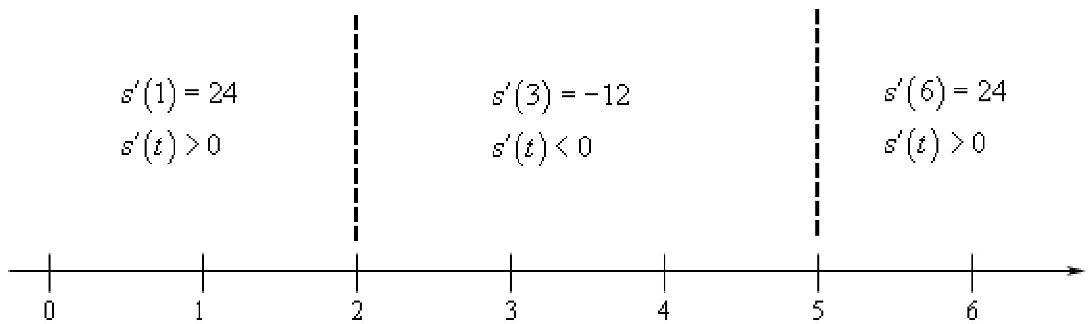
Since polynomials are continuous we know from the Intermediate Value Theorem that if the polynomial ever changes sign then it must have first gone through zero. So, if we knew where the derivative was zero we would know the only points where the derivative *might* change sign.

We can see from the factored form of the derivative that the derivative will be zero at $t=2$ and $t=5$. Let's graph these points on a number line.



Now, we can see that these two points divide the number line into three distinct regions. In reach of these regions we **know** that the derivative will be the same sign. Recall the derivative can only change sign at the two points that are used to divide the number line up into the regions.

Therefore, all that we need to do is to check the derivative at a test point in each region and the derivative in that region will have the same sign as the test point. Here is the number line with the test points and results shown.



Here are the intervals in which the derivative is positive and negative.

$$\begin{aligned} \text{positive}: &\quad -\infty < t < 2 \quad \& \quad 5 < t < \infty \\ \text{negative}: &\quad 2 < t < 5 \end{aligned}$$

We included negative *t*'s here because we could even though they may not make much sense for this problem. Once we know this we also can answer the question. The object is moving to the right and left in the following intervals.

$$\begin{aligned} \text{moving to the right}: &\quad -\infty < t < 2 \quad \& \quad 5 < t < \infty \\ \text{moving to the left}\ \ : &\quad 2 < t < 5 \end{aligned}$$

Make sure that you can do the kind of work that we just did in this example. You will be asked numerous times over the course of the next two chapters to determine where functions are positive and/or negative. If you need some review or want to practice these kinds of problems you should check out the Solving Inequalities section of my Algebra/Trig Review.

Product and Quotient Rule

In the previous section we noted that we had to be careful when differentiating products or quotients. It's now time to look at products and quotients and see why.

First let's take a look at why we have to be careful with products and quotients. Suppose that we have the two functions $f(x)=x^3$ and $g(x)=x^6$. Let's start by computing the derivative of the product of these two functions. This is easy enough to do directly.

$$(f\,g)' = \left(x^3x^6\right)' = \left(x^9\right)' = 9x^8$$

Remember that on occasion we will drop the *(x)* part on the functions to simplify notation somewhat. We've done that in the work above.

Now, let's try the following.

$$f'(x)\,g'(x) = \left(3x^2\right)\left(6x^5\right) = 18x^7$$

So, we can very quickly see that.

$$\left(f\,g\right)' \neq f'\,g'$$

In other words, the derivative of a product is not the product of the derivatives.

Using the same functions we can do the same thing for quotients.

$$\left(\frac{f}{g}\right)' = \left(\frac{x^3}{x^6}\right)' = \left(\frac{1}{x^3}\right)' = \left(x^{-3}\right)' = -3x^{-4} = -\frac{3}{x^4}$$

$$\frac{f'(x)}{g'(x)} = \frac{3x^2}{6x^5} = \frac{1}{2x^3}$$

So, again we can see that,

$$\left(\frac{f}{g}\right)' \neq \frac{f'}{g'}$$

To differentiate products and quotients we have the **Product Rule** and the **Quotient Rule**.

Product Rule

If the two functions *f(x)* and *g(x)* are differentiable (the derivative exist) then the product is differentiable and,

$$\left(f\,g\right)' = f'\,g + f\,g'$$

The proof of this rule is shown in the Extras at the end of this document.

Quotient Rule

If the two functions *f(x)* and *g(x)* are differentiable (the derivative exist) then the quotient is differentiable and,

$$\left(\frac{f}{g}\right)' = \frac{f'\,g - f\,g'}{g^2}$$

The proof of this rule is also shown in the Extras at the end of this document.

Note that the numerator of the quotient rule is very similar to the product rule so be careful to not mix the two up!

Let's do a couple of examples.

Example 1 Differentiate each of the following functions.

(a) $y = \sqrt[3]{x^2}\left(2x - x^2\right)$

(b) $f\left(x\right) = \left(6x^3 - x\right)\left(10 - 20x\right)$

Solution

At this point there really aren't a lot of reasons to use the product rule. As we noted in the previous section all we would need to do for either of these is to just multiply out the product and then differentiate. In fact, we did that for the first one in the previous section.

With that said we will use the product rule on these so we can see an example or two. As we add more functions to our repertoire and as the functions become more complicated the product rule will become more useful and in many cases required.

(a) There's not really a lot to do here other than use the product rule. However, before doing that we should convert the radical to a fractional exponent as always.

$$y = x^{\frac{2}{3}}\left(2x - x^2\right)$$

Now let's take the derivative. So we take the derivative of the first function times the second then add on to that the first function times the derivative of the second function.

$$y' = \frac{2}{3}x^{-\frac{1}{3}}\left(2x - x^2\right) + x^{\frac{2}{3}}\left(2 - 2x\right)$$

This is NOT what we got in the previous section for this derivative. However, with some simplification we can arrive at the same answer.

$$\begin{aligned} y' &= \frac{4}{3}x^{\frac{2}{3}} - \frac{2}{3}x^{\frac{5}{3}} + 2x^{\frac{2}{3}} - 2x^{\frac{5}{3}} \\ &= \frac{10}{3}x^{\frac{2}{3}} - \frac{8}{3}x^{\frac{5}{3}} \end{aligned}$$

This is what we got for an answer in the previous section so that is a good check of the product rule.

(b) This one is actually easier than the previous one. Let's just run it through the product rule.

$$\begin{aligned} f'\left(x\right) &= \left(18x^2 - 1\right)\left(10 - 20x\right) + \left(6x^3 - x\right)\left(-20\right) \\ &= -480x^3 + 180x^2 + 40x - 10 \end{aligned}$$

Since it was easy to do we went ahead and simplified the results a little.

Let's now work an example or two with the quotient rule. In this case, unlike the product rule examples, we are going to be able to do some problems that we weren't able to do prior to this point.

Example 2 Differentiate each of the following functions.

(a) $W\left(z\right)=\dfrac{3z+9}{2-z}$

(b) $h\left(x\right)=\dfrac{4\sqrt{x}}{x^2-2}$

(c) $f\left(x\right)=\dfrac{4}{x^6}$

(d) $y=\dfrac{w^6}{5}$

Solution
There's not much to do with these other than use the quotient rule on them.
(a)

$$\begin{aligned}W'\left(z\right)&=\frac{3\left(2-z\right)-\left(3z+9\right)\left(-1\right)}{\left(2-z\right)^2}\\&=\frac{15}{\left(2-z\right)^2}\end{aligned}$$

(b)

$$\begin{aligned}h'\left(x\right)&=\frac{4\left(\frac{1}{2}\right)x^{-\frac{1}{2}}\left(x^2-2\right)-4x^{\frac{1}{2}}\left(2x\right)}{\left(x^2-2\right)^2}\\&=\frac{2x^{\frac{3}{2}}-4x^{-\frac{1}{2}}-8x^{\frac{3}{2}}}{\left(x^2-2\right)^2}\\&=\frac{-6x^{\frac{3}{2}}-4x^{-\frac{1}{2}}}{\left(x^2-2\right)^2}\end{aligned}$$

(c) It seems strange to have this one here rather than being the first example given that it definitely appears to be easier than any of the previous two. It is easier. However, there is a point to doing it here. In this case there are two ways to do compute this derivative. There is an easy way and a hard way and in this case the hard way is the quotient rule. That's the point of this example.

Let's do the quotient rule and see what we get.

$$\begin{aligned}f'\left(x\right)&=\frac{\left(0\right)\left(x^6\right)-4\left(6x^5\right)}{\left(x^6\right)^2}\\&=\frac{-24x^5}{x^{12}}\\&=-\frac{24}{x^7}\end{aligned}$$

Now, that was the "hard" way. So, what was so hard about it? Well actually it wasn't that hard, there is just an easier way to do it that's all. The easy way is to do what we did in the previous section.

$$\begin{aligned} f'(x) &= 4x^{-6} \\ &= -24x^{-7} \\ &= -\frac{24}{x^7} \end{aligned}$$

Either way will work, but I'd rather take the easier route if I had the choice.

(d) This problem also seems a little out of place. However, it is here again to make a point. Do not confuse this with a quotient rule problem. There is no reason to use the quotient rule on this. Simply rewrite the function as

$$y = \frac{1}{5}w^6$$

and differentiate as always.

$$y' = \frac{6}{5}w^5$$

Example 3 Suppose that the amount of air in a balloon at any time t is given by

$$V(t) = \frac{6\sqrt[3]{t}}{4t+1}$$

Determine if the balloon is being filled with air or being drained of air at $t=8$.

Solution

If the balloon is being filled with air then the volume is increasing and if it's being drained of air then the volume will be decreasing. In other words, we need to get the derivative so that we can determine the rate of change of the volume at $t=8$.

This will require the quotient rule.

$$\begin{aligned} V'(t) &= \frac{2t^{-\frac{2}{3}}(4t+1) - 6t^{\frac{1}{3}}(4)}{(4t+1)^2} \\ &= \frac{-16t^{\frac{1}{3}} + 2t^{-\frac{2}{3}}}{(4t+1)^2} \\ &= \frac{-16t^{\frac{1}{3}} + \frac{2}{t^{\frac{2}{3}}}}{(4t+1)^2} \end{aligned}$$

Note that we simplified the numerator more than usual here. This was only done to make the derivative easier to evaluate.

The rate of change of the volume at $t=8$ is then,

$$V'(8) = \frac{-16(2)+\frac{2}{4}}{(33)^2} \qquad (8)^{\frac{1}{3}} = 2 \qquad (8)^{\frac{2}{3}} = \left((8)^{\frac{1}{3}}\right)^2 = (2)^2 = 4$$
$$= -\frac{63}{2178}$$

So, the rate of change of the volume at $t=8$ is negative and so the volume must be decreasing. Therefore air is being drained out of the balloon at $t=8$.

Note that the product rule can be extended to more than two functions, for instance.

$$(f\,g\,h)' = f'\,g\,h + f\,g'\,h + f\,g\,h'$$
$$(f\,g\,h\,w)' = f'\,g\,h\,w + f\,g'\,h\,w + f\,g\,h'\,w + f\,g\,h\,w'$$

With this section and the previous section we are now able to differentiate powers of x as well as sums, differences, products and quotients of these kinds of functions. However, there are many more functions out there in the world that are not in this form. The next few sections give many of these functions as well as give their derivatives.

Derivatives of Trig Functions

In this section we are going to take a look at the derivatives of the six trig functions. Two of the derivatives will be derived. The remaining four are left to the reader and will follow similar proofs for the two given here.

We'll start with the derivative of the sine function. To do this we will need to use the definition of the derivative.

$$\frac{d}{dx}(\sin(x)) = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$$

Since we can't just plug in $h=0$ to evaluate the limit we will need to use a trig formula on the first sine in the numerator. Doing this gives us,

$$\begin{aligned}\frac{d}{dx}(\sin(x)) &= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h} \\ &= \lim_{h\to 0}\frac{\sin(x)(\cos(h)-1)+\cos(x)\sin(h)}{h} \\ &= \lim_{h\to 0}\sin(x)\frac{\cos(h)-1}{h} + \lim_{h\to 0}\cos(x)\frac{\sin(h)}{h}\end{aligned}$$

As you can see upon using the trig formula we can combine the first and third term. We can then break up the fraction into two pieces, both of which can be dealt with separately.

Now, both of the limits here are limits as h approaches zero. In the first limit we have a $\sin(x)$ and in the second limit we have a $\cos(x)$. Both of these are only functions of x and as h moves in towards zero this has no affect on the value of x. Therefore, as far as the limits are concerned, these two functions are constants and can be factored out of their respective limits. Doing this gives,

$$\frac{d}{dx}\left(\sin\left(x\right)\right)=\sin\left(x\right)\lim_{h\to 0}\frac{\cos\left(h\right)-1}{h}+\cos\left(x\right)\lim_{h\to 0}\frac{\sin\left(h\right)}{h}$$

To finish off this proof we will need the following facts

Fact

$$\lim_{h\to 0}\frac{\cos\left(h\right)-1}{h}=0 \qquad\qquad \lim_{h\to 0}\frac{\sin\left(h\right)}{h}=1$$

We will eventually be able to prove these limits. Ironically, the proof of these limits will be an application of the derivative that we'll be taking a look at in the next chapter.

Using these limits gives us the final answer for the derivative of the sine function.

$$\frac{d}{dx}\left(\sin\left(x\right)\right)=\cos\left(x\right)$$

Differentiating cosine is done in a similar fashion. It will require a different trig formula, but other than that is an almost identical proof. The details will be left to you. When done with the proof you should get,

$$\frac{d}{dx}\left(\cos\left(x\right)\right)=-\sin\left(x\right)$$

With these two out of the way the remaining four are fairly simple to get. All the remaining four trig functions can be defined in terms of sine and cosine and these definitions can be used to get their derivatives.

Let's take a look at tangent. Tangent is defined as,

$$\tan\left(x\right)=\frac{\sin\left(x\right)}{\cos\left(x\right)}$$

Now that we have the derivatives of sine and cosine all that we need to do is use the quotient rule on this. Let's do that.

$$\begin{aligned}\frac{d}{dx}\left(\tan(x)\right) &= \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right)\\ &= \frac{\cos(x)\cos(x)-\sin(x)\left(-\sin(x)\right)}{\left(\cos(x)\right)^2}\\ &= \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)}\\ &= \frac{1}{\cos^2(x)}\end{aligned}$$

Don't forget that $\cos^2(x)+\sin^2(x)=1$! Now, while this is a formula for the derivative we can go one step farther by recalling the definition of secant.

$$\begin{aligned}\frac{d}{dx}\left(\tan(x)\right) &= \frac{1}{\cos^2(x)}\\ &= \sec^2(x)\end{aligned}$$

The remaining three trig functions are also quotients involving sine and cosine and so can be differentiated in a similar manner. We'll leave the details to you. Here are the derivatives of all six of the trig functions.

Derivatives of the six trig functions

$$\frac{d}{dx}\left(\sin(x)\right)=\cos(x) \qquad \frac{d}{dx}\left(\cos(x)\right)=-\sin(x)$$

$$\frac{d}{dx}\left(\tan(x)\right)=\sec^2(x) \qquad \frac{d}{dx}\left(\cot(x)\right)=-\csc^2(x)$$

$$\frac{d}{dx}\left(\sec(x)\right)=\sec(x)\tan(x) \qquad \frac{d}{dx}\left(\csc(x)\right)=-\csc(x)\cot(x)$$

Let's work some examples.

Example 1 Differentiate each of the following functions.

(a) $g(x)=3\sec(x)-10\cos(x)$

(b) $h(w)=3w^{-4}-w^2\tan(w)$

(c) $y=5\sin(x)\cot(x)+4\csc(x)$

(d) $P(t)=\dfrac{\sin(t)}{3-2\cos(t)}$

Solution

(a) There really isn't a whole lot to this problem. We'll just differentiate each term using the formulas from above.

$$g'(x) = 3\sec(x)\tan(x) - 10(-\sin(x))$$
$$= 3\sec(x)\tan(x) + 10\sin(x)$$

(b) In this part we will need to use the product rule on the second term and note that we really will need the product rule here. There is no other way to do this derivative unlike with polynomials where we can just multiply them out.

We will also need to be careful with the minus sign in front of the second term and make sure that it gets dealt with properly. The easiest way to make sure the minus sign does get dealt with properly is to think of it as part of the first function in the product.

The derivative of this function is then.

$$h'(w) = -12w^{-5} - 2w\tan(w) - w^2\sec^2(w)$$

(c) As with the previous part we'll need to use the product rule on the first term. We will also think of the 5 as part of the first function in the product to make sure we deal with it correctly.

Here's the derivative of this function.

$$y' = 5\cos(x)\cot(x) + 5\sin(x)\left(-\csc^2(x)\right) - 4\csc(x)\cot(x)$$
$$= 5\cos(x)\cot(x) - 5\csc(x) - 4\csc(x)\cot(x)$$

Note that in the simplification step we took advantage of the fact that

$$\csc(x) = \frac{1}{\sin(x)}$$

to simplify the second term a little.

(d) In this part we'll need to use the quotient rule.

$$P'(t) = \frac{\cos(t)\left(3 - 2\cos(t)\right) - \sin(t)\left(2\sin(t)\right)}{\left(3 - 2\cos(t)\right)^2}$$
$$= \frac{3\cos(t) - 2\cos^2(t) - 2\sin^2(t)}{\left(3 - 2\cos(t)\right)^2}$$

Be careful with the signs when differentiating the denominator. The negative sign we get from differentiating the cosine will cancel against the negative sign that is already there.

This appears to be done, but there is actually a fair amount of simplification that can yet be done. To do this we need to factor out a "-2" from the last two terms in the numerator and the make use of the fact that $\cos^2(\theta) + \sin^2(\theta) = 1$.

$$P'(t) = \frac{3\cos(t) - 2\left(\cos^2(t) + \sin^2(t)\right)}{\left(3 - 2\cos(t)\right)^2}$$
$$= \frac{3\cos(t) - 2}{\left(3 - 2\cos(t)\right)^2}$$

Example 2 Suppose that the amount of money in a bank account is given by
$$P(t) = 500 + 100\cos(t) - 100\sin(t)$$
where t is in years. During the first 10 years in which the account is open when is the amount of money in the account increasing?

Solution
To determine when the amount of money is increasing we need to determine when the rate of change is positive. Since we know that the rate of change is given by the derivative that is the first thing that we need to find.

$$P'(t) = -100\sin(t) - 100\cos(t)$$

Now, we need to determine where in the first 10 years this will be positive. This is equivalent to asking where in the interval [0, 10] is the derivative positive. Recall that both sine and cosine are continuous functions and so the derivative is also a continuous function. The Intermediate Value Theorem then tells us that the derivative can only change sign if it first goes through zero.

So, we need to solve the following equation.
$$-100\sin(t) - 100\cos(t) = 0$$
$$100\sin(t) = -100\cos(t)$$
$$\frac{\sin(t)}{\cos(t)} = -1$$
$$\tan(t) = -1$$

The solution to this equation is,
$$t = \frac{3\pi}{4} + 2\pi n, \qquad n = 0, \pm1, \pm2, \ldots$$
$$t = \frac{7\pi}{4} + 2\pi n, \qquad n = 0, \pm1, \pm2, \ldots$$

If you don't recall how to solve trig equations go back and take a look at the section on solving trig equations in the Review chapter.

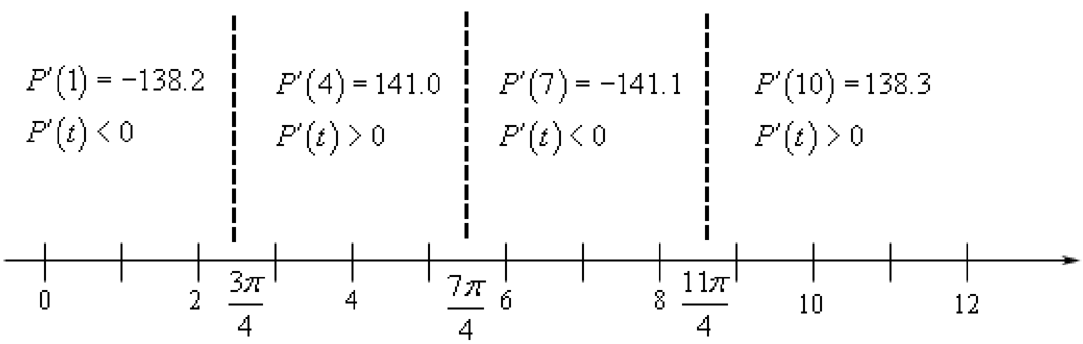
We are only interested in those solutions that fall in the range [0, 10]. Those solutions are,

$$t = \frac{3\pi}{4} = 2.3561 \qquad t = \frac{3\pi}{4} + 2\pi = \frac{11\pi}{4} = 8.6394$$

$$t = \frac{7\pi}{4} = 5.4978$$

So, much like solving polynomial inequalities all that we need to do is sketch in a number line and add in these points. These points will divide the number line into regions in which the derivative must always be the same sign. All that we need to do then is choose a test point from each region to determine the sign of the derivative in that region.

Here is the number line with all the information on it.



So, it looks like the amount of money in the bank account will be increasing during the following intervals.

$$\frac{3\pi}{4} < t < \frac{7\pi}{4} \qquad \frac{11\pi}{4} < t < 10$$

Note that we can't say anything about what is happening after $t=10$ since we haven't done any work for t's after that point.

So, in this section we saw how to differentiate trig functions. We also saw in the last example that our interpretations of the derivative are still valid.

Also, it is important that we be able to solve trig equations as this is something that will arise off and on in this course. It is also important that we can do the kinds of number lines that we used in the last example to determine where a function is positive and where a function is negative. This is something that we will be doing on occasion in both this chapter and the next.

Derivatives of Exponential and Logarithm Functions

The next set of functions that we want to take a look at are exponential and logarithm functions. The most common exponential and logarithm functions in a calculus course are the natural exponential function, $\mathbf{e}^x$, and the natural logarithm function, $\ln(x)$. We will take a more general approach however and look at the general exponential and logarithm function.

Exponential Functions

We'll start off by looking at the exponential function,

$$f(x) = a^x$$

We want to differentiate this. The power rule that we looked at a couple of sections ago won't work as that required the exponent to be a fixed number and the base to be a variable. That is exactly the opposite from what we've got with this function. So, we're going to have to start with the definition of the derivative.

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{a^{x+h}-a^x}{h} \\ &= \lim_{h\to 0}\frac{a^x a^h-a^x}{h} \\ &= \lim_{h\to 0}\frac{a^x\left(a^h-1\right)}{h} \end{aligned}$$

Now, the a^x is not affected by the limit since it doesn't have any h's in it and so is a constant as far as the limit is concerned. We can therefore factor this out of the limit. This gives,

$$f'(x) = a^x \lim_{h\to 0}\frac{a^h-1}{h}$$

Now let's notice that the limit is exactly the definition of the derivative at $x=0$. Therefore, the derivative becomes,

$$f'(x) = f'(0)\,a^x$$

So, we are kind of stuck we need to know the derivative in order to get the derivative! The following fact will help with one possible value of a.

Fact

For the natural exponential function, $f(x) = \mathbf{e}^x$ we have $f'(0) = 1$.

So, provided we are using the natural exponential function we get the following.

$$f(x) = \mathbf{e}^x \qquad \Rightarrow \qquad f'(x) = \mathbf{e}^x$$

At this point we're missing some knowledge that will allow us to easily get the derivative for a general function. Eventually we will be able to show that for a general exponential function we have,

$$f(x) = a^x \qquad \Rightarrow \qquad f'(x) = a^x \ln(a)$$

Logarithm Functions

In this case we will need to start with the following fact about functions that are inverses of each other.

Fact

If *f(x)* and *g(x)* are inverses of each other then,

$$g'(x) = \frac{1}{f'(g(x))}$$

So, how is this fact useful to us? Well recall that the natural exponential function and the natural logarithm function are inverses of each other and we know what the derivative of the natural exponential function is!

So, if we have $f(x) = \mathbf{e}^x$ and $g(x) = \ln x$ then,

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\mathbf{e}^{g(x)}} = \frac{1}{\mathbf{e}^{\ln x}} = \frac{1}{x}$$

The last step just uses the fact that the two functions are inverses of each other.

Putting this all together gives,

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \qquad x > 0$$

Note that we need to require that $x>0$ since this is required for the logarithm and so must also be required for its derivative. In can be shown that,

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x} \qquad x \neq 0$$

Using this all we need to avoid is $x=0$.

In this case, unlike the exponential function case, we can actually find the derivative of the general logarithm function. All that we need is the change of base formula. Using the change of base formula we can write a general logarithm as,

$$\log_a x = \frac{\ln x}{\ln a}$$

Differentiation is then fairly simple.

$$\begin{aligned}\frac{d}{dx}\left(\log_a x\right) &= \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right)\\ &= \frac{1}{\ln a}\frac{d}{dx}\left(\ln x\right)\\ &= \frac{1}{x\ln a}\end{aligned}$$

We took advantage of the fact that a was a constant and so $\ln a$ is also a constant and can be factored out of the derivative. Putting all this together gives,

$$\frac{d}{dx}\left(\log_a x\right) = \frac{1}{x\ln a}$$

Here is a summary of the derivatives in this section.

$$\frac{d}{dx}\left(\mathbf{e}^x\right) = \mathbf{e}^x \qquad \frac{d}{dx}\left(a^x\right) = a^x \ln a$$

$$\frac{d}{dx}\left(\ln x\right) = \frac{1}{x} \qquad \frac{d}{dx}\left(\log_a x\right) = \frac{1}{x\ln a}$$

Okay, now that we have the derivations of the formulas out of the way let's compute a couple of derivatives.

Example 1 Differentiate each of the following functions.

(a) $R(w) = 4^w - 5\log_9 w$

(b) $f(x) = 3\mathbf{e}^x + 10x^3 \ln x$

(c) $y = \dfrac{5\mathbf{e}^x}{3\mathbf{e}^x + 1}$

Solution

(a) This will be the only example that doesn't involve the natural exponential and natural logarithm functions.

$$R'(w) = 4^w \ln 4 - \frac{5}{w\ln 9}$$

(b) Not much to this one. Just remember to use the product rule on the second term.

$$\begin{aligned}f'(x) &= 3\mathbf{e}^x + 30x^2 \ln x + 10x^3\left(\frac{1}{x}\right)\\ &= 3\mathbf{e}^x + 30x^2 \ln x + 10x^2\end{aligned}$$

(c) We'll need to use the quotient rule on this one.

$$y = \frac{5\mathbf{e}^x\left(3\mathbf{e}^x+1\right)-\left(5\mathbf{e}^x\right)\left(3\mathbf{e}^x\right)}{\left(3\mathbf{e}^x+1\right)^2}$$
$$= \frac{15\mathbf{e}^{2x}+5\mathbf{e}^x-15\mathbf{e}^{2x}}{\left(3\mathbf{e}^x+1\right)^2}$$
$$= \frac{5\mathbf{e}^x}{\left(3\mathbf{e}^x+1\right)^2}$$

There's really not a lot to differentiating natural logarithms and natural exponential functions as long as you remember the formulas.

Example 2 Suppose that the position of an object is given by
$$s\left(t\right) = t\mathbf{e}^t$$
Does the object ever stop moving?

Solution
First we will need the derivative. We need this to determine if the object ever stops moving since at that point (provided there is one) the velocity will be zero and recall that the derivative is the velocity.

The derivative is,
$$s'\left(t\right) = \mathbf{e}^t + t\mathbf{e}^t = \left(1+t\right)\mathbf{e}^t$$

So, we need to determine if the derivative is ever zero. To do this we will need to solve,
$$\left(1+t\right)\mathbf{e}^t = 0$$

Now, we know that exponential functions are never zero and so this will only be zero at $t=-1$. So, if we are going to allow negative values of t then the object will stop moving once at $t=-1$. If we aren't going to allow negative values of t then the object will never stop moving.

Derivatives of Inverse Trig Functions

In this section we are going to look at the derivatives of the inverse trig functions. In order to derive the derivatives of inverse trig functions we'll need the formula from the last section relating the derivatives of inverse functions.

Since it's going to be so important to our work here we'll go ahead and give it again. If $f(x)$ and $g(x)$ are inverse functions then,
$$g'\left(x\right) = \frac{1}{f'\left(g\left(x\right)\right)}$$

Recall as well that two functions are inverses if *f(g(x))=x* and *g(f(x))=x*.

Inverse Sine

Let's start with inverse sine. Here is the definition of the inverse sine.

$$y = \sin^{-1} x \qquad \Leftrightarrow \qquad \sin y = x \qquad \text{for} \qquad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

So, evaluating an inverse trig function is the same as asking what angle (*i.e.* *y*) did we plug into the sine function to get *x*. Let's work a quick example.

Example 1 Evaluate $\sin^{-1}\left(\frac{1}{2}\right)$

Solution

So we are really asking what angle *y* solves the following equation.

$$\sin\left(y\right) = \frac{1}{2}$$

and we are restricted to the values of *y* above.

From a unit circle we can quickly see that $y = \frac{\pi}{6}$.

Note as well that since $-1 \le \sin\left(y\right) \le 1$ we also have $-1 \le x \le 1$. We also have the following relationship between the inverse sine function and the sine function.

$$\sin\left(\sin^{-1} x\right) = x \qquad\qquad \sin^{-1}\left(\sin x\right) = x$$

In other words they are inverses of each other. This means that we can use the fact above to find the derivative of inverse sine. Let's start with,

$$f\left(x\right) = \sin x \qquad\qquad g\left(x\right) = \sin^{-1} x$$

Then,

$$g'\left(x\right) = \frac{1}{f'\left(g\left(x\right)\right)} = \frac{1}{\cos\left(\sin^{-1} x\right)}$$

This is not a very useful formula. Let's see if we can get a better formula. Let's start by recalling the definition of the inverse sine function.

$$y = \sin^{-1}\left(x\right) \qquad \Rightarrow \qquad x = \sin\left(y\right)$$

Using the first part of this definition the denominator in the derivative becomes,

$$\cos\left(\sin^{-1} x\right) = \cos\left(y\right)$$

Now, recall that

$$\cos^2 y + \sin^2 y = 1 \qquad\qquad \cos y = \sqrt{1-\sin^2 y}$$

Using this, the denominator is now,

$$\cos\left(\sin^{-1} x\right) = \cos\left(y\right) = \sqrt{1-\sin^2 y}$$

Now, use the second part of the definition of the inverse sine function. The denominator is then,

$$\cos\left(\sin^{-1} x\right) = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$$

Putting all of this together gives the following derivative.

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$$

Inverse Cosine

Now let's take a look at the inverse cosine. Here is the definition for the inverse cosine.

$$y = \cos^{-1} x \qquad \Leftrightarrow \qquad \cos y = x \qquad \text{for} \qquad 0 \le y \le \pi$$

Example 2 Evaluate $\cos\left(-\frac{\sqrt{2}}{2}\right)$.

Solution

As with the inverse sine we are really just asking the following.

$$\cos y = -\frac{\sqrt{2}}{2}$$

where y must meet the requirements given above. From a unit circle we can see that we must have $y = \frac{3\pi}{4}$.

We will also have $-1 \le x \le 1$ here as we did the inverse sine and we also have the following facts.

$$\cos\left(\cos^{-1} x\right) = x \qquad\qquad \cos^{-1}\left(\cos x\right) = x$$

Once again they are inverses of each other.

So to find the derivative we'll do the same kind of work that we did with the inverse sine above. If we start with

$$f\left(x\right) = \cos x \qquad\qquad g\left(x\right) = \cos^{-1} x$$

then,

$$g'\left(x\right) = \frac{1}{f'\left(g\left(x\right)\right)} = \frac{1}{-\sin\left(\cos^{-1} x\right)}$$

Simplifying the denominator here is almost identical to the work we did for the inverse sine and so isn't shown here. Upon simplifying we get the following derivative.

$$\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$$

So, the derivative of the inverse cosine is nearly identical to the derivative of the inverse sine.

Inverse Tangent

Here is the definition of the inverse tangent.

$$y = \tan^{-1} x \qquad \Leftrightarrow \qquad \tan y = x \qquad \text{for} \qquad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

Notice that we can't let *y* be either of the two endpoints in the restriction above since tangent isn't even defined at those two points.

Example 3 Evaluate $\tan^{-1} 1$

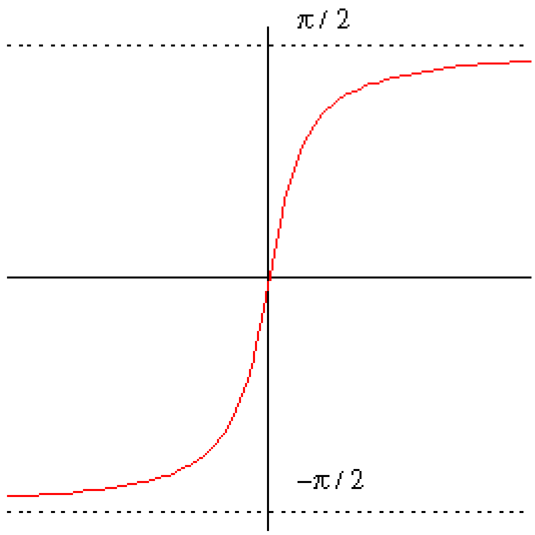
Solution

Here we are asking,

$$\tan y = 1$$

where *y* satisfies the restrictions given above. From a unit circle we can see that $y = \frac{\pi}{4}$.

In this case, unlike the previous two we have no restriction on *x*. So, we can plug any *x* into the inverse trig function. This means that we can ask for the limits of the inverse tangent function as *x* goes to plus or minus infinity. To do this we'll need the graph of the inverse tangent function. This is shown below.



From this graph we can see that

$$\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2} \qquad\qquad \lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2}$$

The tangent and inverse tangent functions are inverse functions so,

$$\tan\left(\tan^{-1} x\right) = x \qquad\qquad \tan^{-1}\left(\tan x\right) = x$$

Therefore to find the derivative of the inverse tangent function we can start with

$$f(x) = \tan x \qquad\qquad g(x) = \tan^{-1} x$$

We then have,

$$g'(x) = \frac{1}{f'\left(g(x)\right)} = \frac{1}{\sec^2\left(\tan^{-1} x\right)}$$

Simplifying the denominator is similar to the inverse sine, but different enough to warrant showing the details. We'll start with the definition of the inverse tangent.

$$y = \tan^{-1} x \qquad\qquad \Rightarrow \qquad\qquad \tan y = x$$

The denominator is then,

$$\sec^2\left(\tan^{-1} x\right) = \sec^2 y$$

Now, if we start with the fact that

$$\cos^2 y + \sin^2 y = 1$$

and divide every term by $\cos^2 y$ we will get,

$$1 + \tan^2 y = \sec^2 y$$

The denominator is then,

$$\sec^2\left(\tan^{-1} x\right) = \sec^2 y = 1 + \tan^2 y$$

Finally using the second portion of the definition of the inverse tangent function gives us,

$$\sec^2\left(\tan^{-1} x\right) = 1 + \tan^2 y = 1 + x^2$$

The derivative of the inverse tangent is then,

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

Summary

There are three more inverse trig functions but the three shown here the most common ones. Formulas for the remaining three could be derived by a similar process as we did those above. Here are the derivatives of all six inverse trig functions.

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2} \qquad \frac{d}{dx}\left(\cot^{-1} x\right) = -\frac{1}{1+x^2}$$
$$\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}\left(\csc^{-1} x\right) = -\frac{1}{x\sqrt{x^2-1}}$$

Example 4 Differentiate the following functions.

(a) $f\left(t\right) = 4\cos^{-1}\left(x\right) - 10\tan^{-1}\left(t\right)$

(b) $y = \sqrt{z}\sin^{-1}\left(z\right)$

Solution

(a) Not much to do with this one other than differentiate each term.

$$f'\left(t\right) = -\frac{4}{\sqrt{1-t^2}} - \frac{10}{1+t^2}$$

(b) Don't forget to convert the radical to fractional exponents before using the product rule.

$$y' = \frac{1}{2}z^{-\frac{1}{2}}\sin^{-1}\left(z\right) + \frac{\sqrt{z}}{\sqrt{1-z^2}}$$

Alternate Notation

There is some alternate notation that is used on occasion to denote the inverse trig functions. This notation is,

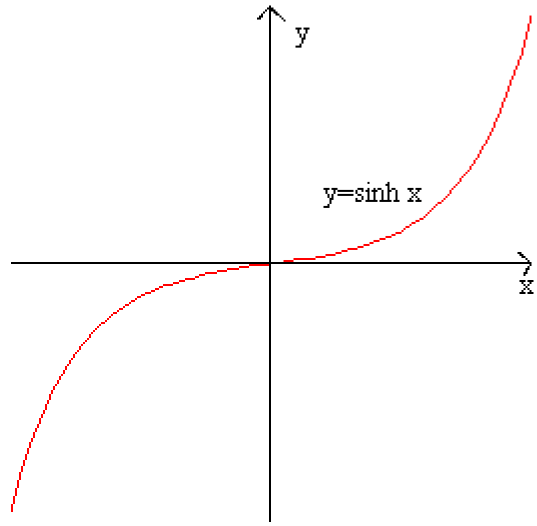
$$\sin^{-1} x = \arcsin x \qquad \cos^{-1} x = \arccos x$$
$$\tan^{-1} x = \arctan x \qquad \cot^{-1} x = \operatorname{arccot} x$$
$$\sec^{-1} x = \operatorname{arcsec} x \qquad \csc^{-1} x = \operatorname{arccsc} x$$

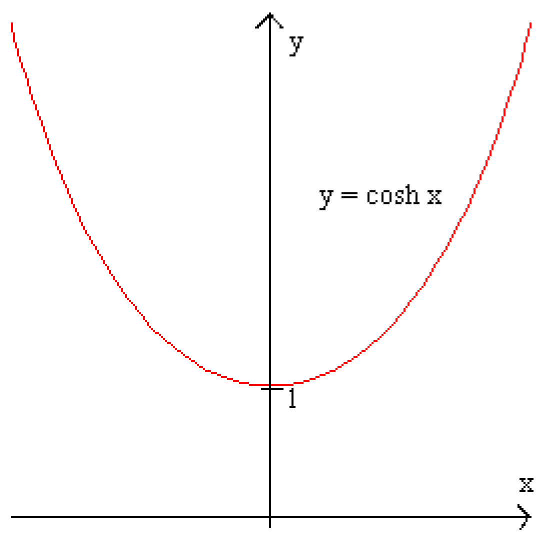
Derivatives of Hyperbolic Trig Functions

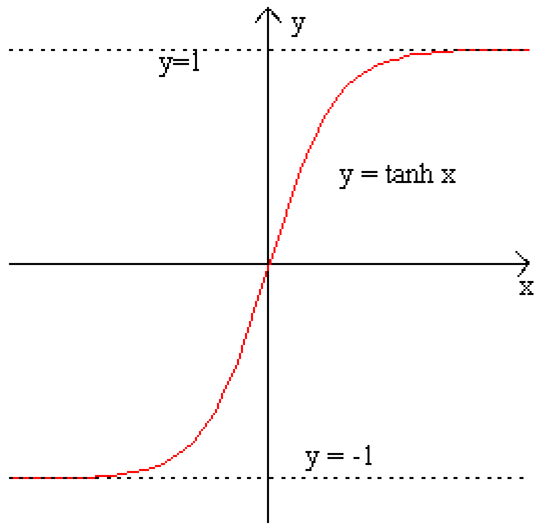
The last set of functions that we're going to be looking at are the hyperbolic trig functions. In many physical situations combinations of $\mathbf{e}^x$ and $\mathbf{e}^{-x}$ arise fairly often. Because of this these combinations are given names. These are the six hyperbolic trig functions. They are defined as follows.

$$\sinh x = \frac{\mathbf{e}^x - \mathbf{e}^{-x}}{2} \qquad \cosh x = \frac{\mathbf{e}^x + \mathbf{e}^{-x}}{2}$$
$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth = \frac{\cosh x}{\sinh x} = \frac{1}{\tanh x}$$
$$\operatorname{sech} x = \frac{1}{\cosh x} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

Here are the graphs of the three main hyperbolic trig functions.







We also have the following facts about the hyperbolic trig functions.

$$\sinh(-x) = -\sinh(x) \qquad \cosh(-x) = \cosh(x)$$
$$\cosh^2(x) - \sinh^2(x) = 1 \qquad 1 - \tanh^2(x) = \operatorname{sech}^2(x)$$

You'll note that these are similar, but not quite the same, to some of the more common trig identities.

Because the hyperbolic trig functions are defined in terms of exponential functions finding their derivatives is fairly simple. We'll do the derivative for hyperbolic sine and leave the rest to you as an exercise.

$$\begin{aligned}\frac{d}{dx}(\sinh x) &= \frac{d}{dx}\left(\frac{\mathbf{e}^{x} - \mathbf{e}^{-x}}{2}\right)\\ &= \frac{\mathbf{e}^{x} + \mathbf{e}^{-x}}{2}\\ &= \cosh x\end{aligned}$$

For the rest we can either use the definition or use the quotient rule. Here are all six derivatives.

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\cosh x) = \sinh x$$
$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$
$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

Example 1 Differentiate each of the following functions.

(a) $f(x) = 2x^5 \cosh x$

(b) $h(t) = \dfrac{\sinh t}{t+1}$

Solution

(a)

$$f'(x) = 10x^4 \cosh x + 2x^5 \sinh x$$

(b)

$$h'(t) = \frac{(t+1)\cosh t - \sinh t}{(t+1)^2}$$

Chain Rule

We've taken a lot of derivatives over the course of the last six sections. However, if you look back they have all been functions similar to the following kinds of functions.

$$R(z) = \sqrt{z} \qquad f(t) = t^{50} \qquad y = \tan(x) \qquad h(w) = \mathbf{e}^w \qquad g(x) = \ln x$$

These are all fairly simple functions. What about functions like the following,

$$R(z) = \sqrt{5z-8} \qquad f(t) = \left(2t^3 + \cos(t)\right)^{50} \qquad y = \tan\left(\sqrt[3]{3x^2} + \tan(5x)\right)$$
$$h(w) = \mathbf{e}^{w^4 - 3w^2 + 9} \qquad g(x) = \ln\left(x^{-4} + x^4\right)$$

None of our rules will work on these functions and yet some of these functions are closer to the derivatives that we're liable to run into than the functions in the first set.

Let's take the first one for example. Back in the section on the definition of the derivative we actually used the definition to compute this derivative. In that section we found that,

$$R'(z) = \frac{5}{2\sqrt{5z-8}}$$

If we were to just use the power rule on this we would get,

$$\frac{1}{2}\left(5z-8\right)^{-\frac{1}{2}} = \frac{1}{2\sqrt{5z-8}}$$

which is not the derivative that we computed using the definition. It is close, but it's not the same. So, the power rule alone simply won't work to get the derivative of this function.

Let's keep looking at this function and note that if we define,

$$f\left(z\right) = \sqrt{z} \qquad\qquad g\left(z\right) = 5z-8$$

then we can write the function as a composition.

$$R\left(z\right) = \left(f \circ g\right)\left(z\right) = f\left(g\left(z\right)\right) = \sqrt{5z-8}$$

It turns out that it's actually fairly simple to differentiate a function composition using the **Chain Rule**. There are two forms of the chain rule. Here they are.

Chain Rule

Suppose that we have two functions *f(x)* and *g(x)* and they are both differentiable.

1. If we define $F\left(x\right) = \left(f \circ g\right)\left(x\right)$ then the derivative of *F(x)* is,

$$F'\left(x\right) = f'\left(g\left(x\right)\right)\; g'\left(x\right)$$

2. If we have *y=f(u)* and *u=g(x)* then the derivative of *y* is,

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}$$

Each of these forms have their uses, however we will work mostly with the first form in this class.

Example 1 Use the Chain Rule to differentiate $R\left(z\right) = \sqrt{5z-8}$.

Solution

We've already identified the two functions that we needed for the composition, but let's write them back down anyway and take their derivatives.

$$f\left(z\right) = \sqrt{z} \qquad\qquad g\left(z\right) = 5z-8$$

$$f'\left(z\right) = \frac{1}{2\sqrt{z}} \qquad\qquad g'\left(z\right) = 5$$

So, using the chain rule we get,

$$\begin{aligned} R'(z) &= f'(g(z))\, g'(z) \\ &= f'(5z-8)\, g'(z) \\ &= \frac{1}{2}(5z-8)^{-\frac{1}{2}}(5) \\ &= \frac{1}{2\sqrt{5z-8}}(5) \\ &= \frac{5}{2\sqrt{5z-8}} \end{aligned}$$

And this is what we got using the definition of the derivative.

There is a quick and easy way of remembering the chain rule that doesn't require us to think in terms of function composition. Let's take the function from the previous example and rewrite it slightly.

$$R(z) = \underbrace{(5z-8)}_{\text{inside function}}{}^{\underbrace{\frac{1}{2}}_{\text{outside function}}}$$

This function has an "inside function" and an "outside function". The outside function is the square root or the exponent of $\frac{1}{2}$ depending on how you want to think of it and the inside function is the stuff that we're taking the square root of or raising to the $\frac{1}{2}$, again depending on how you want to look at it.

The derivative is then,

$$R'(z) = \overbrace{\frac{1}{2}\underbrace{(5z-8)}_{\substack{\text{inside function}\\ \text{left alone}}}{}^{-\frac{1}{2}}}^{\substack{\text{derivative of}\\ \text{outside function}}} \underbrace{(5)}_{\substack{\text{derivative of}\\ \text{inside function}}}$$

In general this is how we think of the chain rule. We identify the "inside function" and the "outside function". We then we differentiate the outside function leaving the inside function alone and multiply all of this by the derivative of the inside function. In its general form this is,

$$F'(x) = \underbrace{f'}_{\substack{\text{derivative of}\\ \text{outside function}}} \underbrace{(g(x))}_{\substack{\text{inside function}\\ \text{left alone}}} \underbrace{g'(x)}_{\substack{\text{times derivative}\\ \text{of inside function}}}$$

We can always identify the "outside function" in the examples below by asking our selves how we would evaluate the function. For instance in the *R(z)* case if we were to ask ourselves what *R(2)* is we would first evaluate the stuff under the radical and then finally take the square root of this result. The square root is the last operation that we perform in the evaluation and this is also the outside function. The outside function will always be the last operation you would perform if you were going to evaluate the function.

Let's take a look at some examples.

Example 2 Differentiate each of the following.

(a) $f(x)=\sin\left(3x^2+x\right)$

(b) $f(t)=\left(2t^3+\cos(t)\right)^{50}$

(c) $h(w)=\mathbf{e}^{w^4-3w^2+9}$

(d) $g(x)=\ln\left(x^{-4}+x^4\right)$

(e) $y=\sec(1-5x)$

(f) $P(t)=\cos^4(t)+\cos\left(t^4\right)$

Solution

(a) It looks like the outside function is the sine and the inside function is $3x^2+x$. The derivative is then.

$$f'(x)=\underbrace{\cos}_{\substack{\text{derivative of}\\\text{outside function}}}\quad\underbrace{\left(3x^2+x\right)}_{\substack{\text{leave inside}\\\text{function alone}}}\quad\underbrace{(6x+1)}_{\substack{\text{times derivative}\\\text{of inside function}}}$$

Or,

$$f'(x)=(6x+1)\cos\left(3x^2+x\right)$$

(b) In this case the outside function is the exponent of 50 and the inside function is all the stuff on the inside of the parenthesis. The derivative is then.

$$\begin{aligned}f'(t)&=50\left(2t^3+\cos(t)\right)^{49}\left(6t^2-\sin(t)\right)\\&=50\left(6t^2-\sin(t)\right)\left(2t^3+\cos(t)\right)^{49}\end{aligned}$$

(c) Identifying the outside function in the previous two was fairly simple since it really was the "outside" function in some sense. In this case we need to be a little careful. Recall that the outside function is the last operation that we would perform in an evaluation. In this case if we were to evaluate this function the last operation would be the exponential. Therefore the outside function is the exponential function and the inside function is its exponent.

Here's the derivative.

$$\begin{aligned}h'(w)&=\mathbf{e}^{w^4-3w^2+9}\left(4w^3-6w\right)\\&=\left(4w^3-6w\right)\mathbf{e}^{w^4-3w^2+9}\end{aligned}$$

Remember, we leave the inside function alone when we differentiate the outside function.

So, the derivative of the exponential function (with the inside left alone) is just the original function.

(d) Here the outside function is the natural logarithm and the inside function is stuff on the inside of the logarithm.

$$\begin{aligned} g'(x) &= \frac{1}{x^{-4}+x^4}\left(-4x^{-5}+4x^3\right) \\ &= \frac{-4x^{-5}+4x^3}{x^{-4}+x^4} \end{aligned}$$

Again remember to leave the inside function along when differentiating the outside function. So, upon differentiating the logarithm we end up not with $1/x$ but instead with 1/(inside function).

(e) In this case the outside function is the secant and the inside is the *1-5x*.

$$\begin{aligned} y' &= \sec(1-5x)\tan(1-5x)(-5) \\ &= -5\sec(1-5x)\tan(1-5x) \end{aligned}$$

In this case the derivative of the outside function is sec*(x)*tan*(x)*. However, since we leave the inside function alone we don't get x's in both. Instead we get *1-5x* in both.

(f) There are two points to this problem. First, there are two terms and each will require a different application of the chain rule. That will often be the case. Second, we need to be very careful in choosing the outside and inside function for each term.

Recall that the first term can actually be written as,

$$\cos^4(t) = \left(\cos(t)\right)^4$$

So, in the first term the outside function is the exponent of 4 and the inside function is the cosine. In the second term it's exactly the opposite. In the second term the outside function is the cosine and the inside function is t^4. Here's the derivative for this function.

$$\begin{aligned} P'(t) &= 4\cos^3(t)\left(-\sin(t)\right) - \sin\left(t^4\right)\left(4t^3\right) \\ &= -4\sin(t)\cos^3(t) - 4t^3\sin\left(t^4\right) \end{aligned}$$

There are a couple of general formulas that we can get from some special cases of the chain rule. Let's take a quick look at those.

Example 3 Differentiate each of the following.

(a) $f(x) = \left[g(x)\right]^n$

(b) $f(x) = \mathbf{e}^{g(x)}$

(c) $f(x) = \ln\left(g(x)\right)$

Solution

(a) The outside function is the exponent and the inside is *g(x)*.

$$f'(x) = n\left[g(x)\right]^{n-1} g'(x)$$

(b) The outside function is the exponential function and the inside is *g(x)*.

$$f'(x) = g'(x)\mathbf{e}^{g(x)}$$

(c) The outside function is the logarithm and the inside is *g(x)*.

$$f'(x) = \frac{1}{g(x)} g'(x) = \frac{g'(x)}{g(x)}$$

The formulas in this example are really just special cases of the Chain Rule but may be useful to remember.

Now, let's also not forget the other rules that we've got for doing derivatives.

Example 4 Differentiate each of the following.

(a) $T(x) = \tan^{-1}(2x)\sqrt[3]{1-3x^2}$

(b) $y = \dfrac{\left(x^3+4\right)^5}{\left(1-2x^2\right)^3}$

Solution

(a) This requires the product rule and each derivative in the product rule will require a chain rule application as well.

$$\begin{aligned} T'(x) &= \frac{1}{1+(2x)^2}(2)\left(1-3x^2\right)^{\frac{1}{3}} + \tan^{-1}(2x)\left(\frac{1}{3}\right)\left(1-3x^2\right)^{-\frac{2}{3}}(-6x) \\ &= \frac{2\left(1-3x^2\right)^{\frac{1}{3}}}{1+(2x)^2} - 2x\tan^{-1}(2x)\left(1-3x^2\right)^{-\frac{2}{3}} \end{aligned}$$

(b) In this case we will be using the chain rule in concert with the quotient rule.

$$y' = \frac{5\left(x^3+4\right)^4\left(3x^2\right)\left(1-2x^2\right)^3 - \left(x^3+4\right)^5(3)\left(1-2x^2\right)^2(-4x)}{\left(\left(1-2x^2\right)^3\right)^2}$$

These tend to be a little messy. Notice that when we go to simplify that we'll be able to a fair amount of factoring in the numerator.

$$y' = \frac{\left(x^3+4\right)^4\left(1-2x^2\right)^2\left(5\left(3x^2\right)\left(1-2x^2\right)-\left(x^3+4\right)(3)(-4x)\right)}{\left(1-2x^2\right)^6}$$

$$= \frac{3x\left(x^3+4\right)^4\left(5x-6x^3+16\right)}{\left(1-2x^2\right)^4}$$

Upon factoring, notice that we can cancel some of the terms in the numerator against the denominator. So even though the initial chain rule was fairly messy the final answer is significantly simpler because of the factoring.

The point of this last example is to not forget the other derivative rules that we've got. Most of the examples in this section won't involve the product rule or the quotient rule to make the problems a little shorter. However, in practice they will often be in the same problem.

Now, let's take a look at some slightly more complicated examples.

Example 5 Differentiate each of the following.

(a) $h(z) = \dfrac{2}{\left(4z+\mathbf{e}^{-9z}\right)^{10}}$

(b) $f(y) = \sqrt{2y+\left(3y+4y^2\right)^3}$

(c) $y = \tan\left(\sqrt[3]{3x^2}+\ln\left(5x^4\right)\right)$

(d) $g(t) = \sin^3\left(\mathbf{e}^{1-t}+3\sin(6t)\right)$

Solution

We're going to be a little more careful in these problems than we were in the previous ones. The reason will be quickly apparent.

(a) In this case let's first rewrite the function in a form that we can deal with.

$$h(z) = 2\left(4z+\mathbf{e}^{-9z}\right)^{-10}$$

Now, let's start the derivative.

$$h'(z) = -20\left(4z+\mathbf{e}^{-9z}\right)^{-11}\frac{d}{dz}\left(4z+\mathbf{e}^{-9z}\right)$$

Notice that we didn't actually do the derivative of the inside function yet. This is to allow us to notice that when we do differentiate the second term we will require the chain rule again. Notice as well that we will only need the chain rule on the exponential and not the first term.

In many functions we will be using the chain rule more than once. Let's go ahead and finish this example out.

$$h'(z) = -20\left(4z + \mathbf{e}^{-9z}\right)^{-11}\left(4 - 9\mathbf{e}^{-9z}\right)$$

Be careful with the second application of the chain rule. Only the exponential gets multiplied by the "-9" since that's the derivative of the inside function for that term only. One of the more common mistakes in these kinds of problems is to multiply the whole term by the "-9" and not just the second term.

(b) We'll not put as many words into this example, but we're still going to be careful with this derivative.

$$\begin{aligned} f'(y) &= \frac{1}{2}\left(2y + \left(3y + 4y^2\right)^3\right)^{-\frac{1}{2}} \frac{d}{dy}\left(2y + \left(3y + 4y^2\right)^3\right) \\ &= \frac{1}{2}\left(2y + \left(3y + 4y^2\right)^3\right)^{-\frac{1}{2}} \left(2 + 3\left(3y + 4y^2\right)^2 (3 + 8y)\right) \\ &= \frac{1}{2}\left(2y + \left(3y + 4y^2\right)^3\right)^{-\frac{1}{2}} \left(2 + (9 + 24y)\left(3y + 4y^2\right)^2\right) \end{aligned}$$

As with the first example the second term of the inside function required the chain rule to differentiate it. Also note that again we need to be careful when multiplying by the derivative of the second term.

(c) Let's jump right into this one.

$$\begin{aligned} y' &= \sec^2\left(\sqrt[3]{3x^2} + \ln\left(5x^4\right)\right) \frac{d}{dx}\left(\left(3x^2\right)^{\frac{1}{3}} + \ln\left(5x^4\right)\right) \\ &= \sec^2\left(\sqrt[3]{3x^2} + \ln\left(5x^4\right)\right)\left(\frac{1}{3}\left(3x^2\right)^{-\frac{2}{3}}(6x) + \frac{20x^3}{5x^4}\right) \\ &= \left(2x\left(3x^2\right)^{-\frac{2}{3}} + \frac{4}{x}\right)\sec^2\left(\sqrt[3]{3x^2} + \ln\left(5x^4\right)\right) \end{aligned}$$

In this example both of the terms in the inside function required a separate application of the chain rule.

(d) We'll need to be a little careful with this one.

$$\begin{aligned} g'(t) &= 3\sin^2\left(\mathbf{e}^{1-t} + 3\sin(6t)\right)\frac{d}{dt}\sin\left(\mathbf{e}^{1-t} + 3\sin(6t)\right) \\ &= 3\sin^2\left(\mathbf{e}^{1-t} + 3\sin(6t)\right)\cos\left(\mathbf{e}^{1-t} + 3\sin(6t)\right)\frac{d}{dt}\left(\mathbf{e}^{1-t} + 3\sin(6t)\right) \\ &= 3\sin^2\left(\mathbf{e}^{1-t} + 3\sin(6t)\right)\cos\left(\mathbf{e}^{1-t} + 3\sin(6t)\right)\left(\mathbf{e}^{1-t}(-1) + 3\cos(6t)(6)\right) \\ &= 3\left(-\mathbf{e}^{1-t} + 18\cos(6t)\right)\sin^2\left(\mathbf{e}^{1-t} + 3\sin(6t)\right)\cos\left(\mathbf{e}^{1-t} + 3\sin(6t)\right) \end{aligned}$$

This problem required a total of 4 chain rules to complete.

Sometimes these can get quite unpleasant and require many applications of the chain rule. In these cases it's usually best to be careful as we did in this previous set of examples and write out a couple of extra steps rather than trying to do it all in one step in your head.

Implicit Differentiation

To this point we've done quite a few derivatives, but they have all been derivatives of functions of the form *y=f(x)*. Unfortunately not all the functions that we're going to look at will fall into this form.

Let's take a look at an example of this.

Example 1 Find y' for *xy=1*.

Solution

There are actually two solution methods for this problem.

Solution 1 :

This is the simple way of doing the problem. Just solve for *y* to get the function in the form that we're used to dealing with.

$$y = \frac{1}{x} \qquad \Rightarrow \qquad y' = -\frac{1}{x^2}$$

So, that's easy enough to do. However, there are some functions for which this can't be done. That's where the second solution technique comes into play.

Solution 2 :

In this case we're going to leave the function in the form that we were given. We do want to remember however that we are thinking of *y* as a function of *x*. In other words, *y=y(x)*. Let's rewrite the equation to note this.

$$x\,y\left(x\right) = 1$$

Now, we will differentiate both sides with respect to *x*.

$$\frac{d}{dx}\left(x\,y\left(x\right)\right) = \frac{d}{dx}\left(1\right)$$

The right side is easy. It's just the derivative of a constant. The left side is also easy, but we've got to recognize that we've actually got a product here. So to do the derivative of the left side we'll need to do the product rule.

$$\left(1\right)y\left(x\right) + x\frac{d}{dx}\left(y\right) = 0$$

Now, recall that

$$\frac{d}{dx}(y) = y'$$

so, we get,

$$y + xy' = 0$$

Note that we dropped the *(x)* on the *y* as it's no longer really needed. We just wanted it in the equation to recognize the product rule.

So, just what were we after here? We were after y' and notice that there is now a y' in the equation. So, to get the derivative all that we need to do is solve the equation for y'.

$$y' = -\frac{y}{x}$$

This is not what we got from the first solution. Or at least it doesn't look like the same derivative. Recall however, that we do know what *y* is in terms of *x* and if we plug that in we will get,

$$y' = -\frac{1}{x^2}$$

which is what we got from the first solution.

The process that we used in the second solution to the previous example is called **implicit differentiation**. In the previous example we were able to just solve for *y* and avoid implicit differentiation. However, that won't always be the case.

Let's see an example of this.

Example 2 Find y' for the following function.

$$x^2 + y^2 = 9$$

Solution

This is just a circle and while can solve for *y* this would give,

$$y = \pm\sqrt{9 - x^2}$$

In other words, we would get two functions. This would cause us problems in getting the derivative and so will do us no good. We want a single function for the derivative and using this would give us two.

In this example we really are going to need to do implicit differentiation. We'll do the same thing we did in the first example and remind ourselves that *y* is really a function of *x* and write *y* as *y(x)*.

$$\frac{d}{dx}\left(x^2 + \left[y(x)\right]^2\right) = \frac{d}{dx}(9)$$

$$2x + 2\left[y(x)\right]^1 y'(x) = 0$$

Notice that when we differentiated the y term we used the chain rule. Dropping the *(x)* part of the y and solving for the derivative give,

$$2x + 2yy' = 0$$

$$y' = -\frac{x}{y}$$

Unlike the first example we can't just plug in for y since we wouldn't know which of the two roots to use. Most answers from implicit differentiation will involve both x and y so don't get excited about that when it happens.

We can't forget our interpretations of derivatives.

Example 3 Find the equation of the tangent line to

$$x^2 + y^2 = 9$$

at the point $\left(2, \sqrt{5}\right)$.

Solution

First note that unlike all the other tangent line problems we've done in previous sections we need to be given both the x and the y values of the point. Notice as well that this point does lie on the graph of the circle (you can check by plugging the points into the equation) and so it's okay to talk about the tangent line at this point.

Recall that all we need is the slope of the tangent line and this is nothing more than the derivative evaluated at the point. We've got the derivative from the previous example so,

$$m = y'\Big|_{x=2,\, y=\sqrt{5}} = -\frac{2}{\sqrt{5}}$$

The tangent line is then.

$$y = \sqrt{5} - \frac{2}{\sqrt{5}}\left(x - 2\right)$$

Now, let's work some more examples. In these examples we will no longer write *y(x)* for y. This is just something that we're going to remind ourselves of when we see y. This means that every time we differentiate y we are really going to be using the chain rule.

There is an easy way to remember how to do this. The chain rule really tells us to differentiate the function as we usually would, except we need to add on a derivative of the inside function. In implicit differentiation this means that every time we are differentiating a term with y in it the inside function is the y and we will need to add a y' onto the term since that will be the derivative of the inside function.

Let's see a couple of examples.

Example 4 Find y' for each of the following.

(a) $x^3 y^5 + 3x = 8y^3 + 1$

(b) $x^2 \tan(y) + y^{10} \sec(x) = 2x$

(c) $\mathbf{e}^{2x+3y} + \ln(xy^3) = x^2$

Solution

(a) First differentiate both sides with respect to *x* and notice that the first time on left side will be a product rule.

$$3x^2 y^5 + 5x^3 y^4 y' + 3 = 24y^2 y'$$

Remember that very time we differentiate a *y* we also multiply that term by y' since we are just using the chain rule. Now solve for the derivative.

$$3x^2 y^5 + 3 = 24y^2 y' - 5x^3 y^4 y'$$

$$3x^2 y^5 + 3 = \left(24y^2 - 5x^3 y^4\right) y'$$

$$y' = \frac{3x^2 y^5 + 3}{24y^2 - 5x^3 y^4}$$

The algebra in these can be quite messy so be careful with that.

(b) So, we've got two product rules to deal with this time.

$$2x\tan(y) + x^2 \sec^2(y)\, y' + 10y^9 y' \sec(x) + y^{10}\sec(x)\tan(x) = 2$$

Notice the derivative tacked onto the secant! We differentiated a *y* to get to that point and so we needed to tack a derivative on.

Now, solve for the derivative.

$$\left(x^2 \sec^2(y) + 10y^9 \sec(x)\right) y' = 2 - y^{10}\sec(x)\tan(x) - 2x\tan(y)$$

$$y' = \frac{2 - y^{10}\sec(x)\tan(x) - 2x\tan(y)}{x^2\sec^2(y) + 10y^9\sec(x)}$$

(c) We're going to need to be careful with this problem. We've got a couple chain rules that we're going to need to deal with.

$$\mathbf{e}^{2x+3y}\left(2 + 3y'\right) + \frac{y^3 + 3xy^2 y'}{xy^3} = 2x$$

In both of the chain rules note that the y' didn't get tacked on until we actually differentiated the *y*'s. Now we need to solve for the derivative. This is liable to be somewhat messy.

In order to get the y' on one side we'll need to multiply the exponential through the parenthesis and break up the quotient.

$$2\mathbf{e}^{2x+3y} + 3y'\mathbf{e}^{2x+3y} + \frac{y^3}{xy^3} + \frac{3xy^2 y'}{xy^3} = 2x$$
$$2\mathbf{e}^{2x+3y} + 3y'\mathbf{e}^{2x+3y} + \frac{1}{x} + \frac{3y'}{y} = 2x$$
$$\left(3\mathbf{e}^{2x+3y} + 3y^{-1}\right) y' = 2x - x^{-1} - 2\mathbf{e}^{2x+3y}$$
$$y' = \frac{2x - x^{-1} - 2\mathbf{e}^{2x+3y}}{3\mathbf{e}^{2x+3y} + 3y^{-1}}$$

Note that to make the derivative at least look a little nicer we converted all the fractions to negative exponents.

Okay, we've seen one application of implicit differentiation in the tangent line example above. However, there is another application that we will be seeing in the next section.

In some cases we will have two (or more) functions all of which are functions of a third variable. So, we might have *x(t)* and *y(t)*, for example. In these cases we will be differentiating with respect to *t*. This is just implicit differentiation like we did in the previous examples. Whenever we do a derivative of an *x* we will tack on an x' and whenever we do a derivative of a *y* we will tack on a y'.

Let's take a look at an example of this.

Example 5 Assume that *x=x(t)* and *y=y(t)* and differentiate the following equation with respect to *t*.

$$x^3 y^6 + \mathbf{e}^{1-x} - \cos(5y) = y^2$$

Solution

So, just differentiate as normal and tack on an appropriate derivative at each step. Note as well that the first term will be a product rule.

$$3x^2 x' y^6 + 6x^3 y^5 y' - x'\mathbf{e}^{1-x} + 5y'\sin(5y) = 2yy'$$

There really isn't all that much to this problem. Since there are two derivatives in the problem we won't be bothering to solve for one of them. When we do this kind of problem in the next section the problem will imply which one we need to solve for.

Related Rates

In this section we are going to look at an application of implicit differentiation. Most of the applications of derivatives are in the next chapter however there are a couple of reasons for placing it in this chapter as opposed to putting it into the next chapter with the other applications. The first reason is that it's an application of implicit differentiation and so putting right after that section means that we won't have forgotten how to do implicit differentiation. The other reason is simply that after doing all these derivatives we need to be reminded that there really are actual applications to derivatives. Sometimes it is easy to forget there really is a reason that we're spending all this time on derivatives.

For these related rates problems it's usually best to just jump right into some problems and see how they work.

Example 1 Air is being pumped into a spherical balloon at a rate of 5 cm^3/min. Determine the rate at which the radius of the balloon is increasing when the diameter of the balloon is 20 cm.

Solution
The first thing that we'll need to do here is to identify what information that we've been given and what we want to find. Before we do that let's notice that both the volume of the balloon and the radius of the balloon will vary with time and so are really functions of time.

Now, we know that air is being pumped into the balloon at a rate of 5 cm^3/min. This is the rate at which the volume is increasing. Recall that rates of change are nothing more than derivatives and so we know that,

$$V'(t) = 5$$

We want to determine the rate at which the radius is changing. Again, rates are derivatives and so it looks like we want to determine,

$$r'(t) = ? \qquad \text{when} \quad r = \frac{d}{2} = 10\,\text{cm}$$

Note that we needed to convert the diameter to a radius.

Now that we've identified what we have been given and what we want to find we need to relate these two quantities to each other. In this case we can relate the volume and the radius with the formula for the volume of a sphere.

$$V(t) = \frac{4}{3}\pi\left[r(t)\right]^3$$

We will typically not use the *(t)* part of things, but since this is the first time through one of these we will do that to remind ourselves that they are really functions of *t*.

Now we don't really want a relationship between the volume and the radius. We really want a relationship between their derivatives. We can do this by differentiating both sides with respect to *t*. In other words, we will need to do implicit differentiation.

Doing this gives,

$$V' = 4\pi r^2 r'$$

At this point all that we need to do is plug in what we know and solve for what we want to find.

$$5 = 4\pi\left(10^2\right)r' \qquad \Rightarrow \qquad r' = \frac{1}{80\pi} \text{ cm/min}$$

We can get the units of the derivative be recalling that,

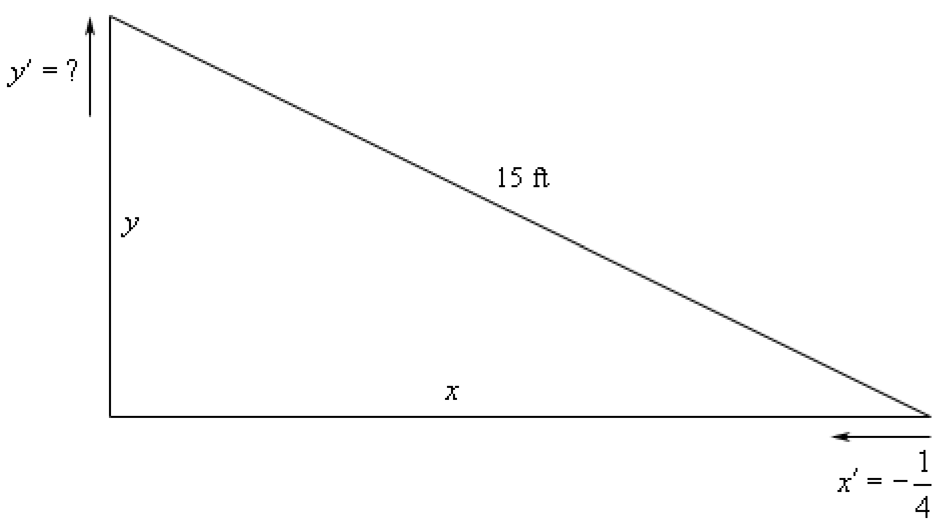
$$r' = \frac{dr}{dt}$$

The units of the derivative will be the units of the numerator (cm in the previous example) divided by the units of the denominator (min in the previous example).

Let's work another example.

Example 2 A 15 foot ladder is resting against the wall. The bottom is initially 10 feet away from the wall and is being pushed towards the wall at a rate of $\frac{1}{4}$ ft/sec. How fast is the top of the ladder moving up the wall 12 seconds after we start pushing?

Solution
The first thing to do in this case is to sketch picture that shows us what is going on.



So, we've defined the distance of the bottom of the latter from the wall to be x and the distance of the top of the ladder from the floor to be y.
We know that the rate at which the bottom of the ladder is moving towards the wall. This is,

$$x' = -\frac{1}{4}$$

Note that the rate is negative since the distance from the wall, x, is decreasing. We always need to be careful with signs with these problems.

We want to find the rate at which the top of the ladder is moving away from the floor. This is y'. Note as well that this quantity should be positive since y will be increasing.

As with the first example we first need a relationship between x and y. We can get this using Pythagorean theorem.

$$x^2 + y^2 = (15)^2 = 225$$

All that we need to do at this point is to differentiate both sides with respect to t and we'll get the relationship between the derivatives.

$$2xx' + 2yy' = 0$$

This equation has two quantities that we don't know and will need to know in order to finish the problem. We need to determine x and y. Both are fairly simple to get. We know that initially $x=10$ and the end is being pushed in towards the wall at a rate of $\frac{1}{4}$ ft/sec and that we are interested in what has happened after 12 seconds. This means that,

$$x = 10 - \frac{1}{4}(12) = 7$$

To find y (after 12 seconds) all that we need to do is reuse the Pythagorean Theorem with the values of x that we just found above.

$$y = \sqrt{225 - x^2} = \sqrt{225 - 49} = \sqrt{176}$$

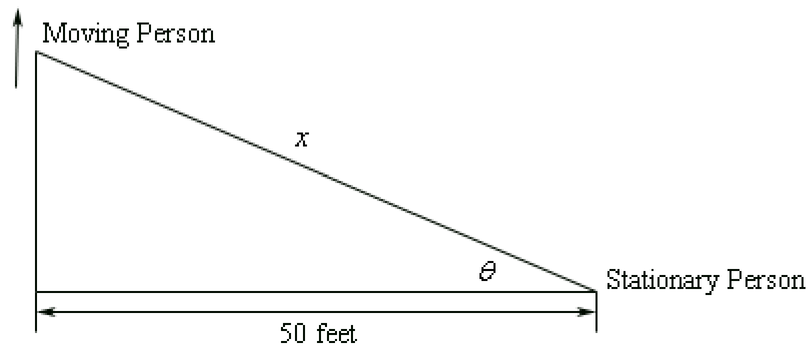
Now all that we need to do is plug into the equation and solve for y'.

$$2(7)\left(-\frac{1}{4}\right) + 2\left(\sqrt{176}\right)y' = 0 \qquad y' = \frac{7/4}{\sqrt{176}} = \frac{7}{4\sqrt{176}} \text{ ft/sec}$$

Notice that we got the correct sign. If we'd gotten a negative then we'd have known that we had made a mistake and we could go back and look for it.

Let's work one more example.

Example 3 Two people are 50 feet apart. One of them starts walking north at a rate so that the angle shown in the diagram below is changing at a constant rate of 0.01 rad/min. At what rate is distance between the two people changing when $\theta = 0.5$ radians?



Solution

This example is not as tricky as it might at first appear. Let's call the distance between them at any point in time *x*. We can then relate all the known quantities by one of two trig formulas.

$$\cos\theta = \frac{50}{x} \qquad\qquad \sec\theta = \frac{x}{50}$$

We want to find x' and we could find *x* if we wanted to at the point in question using the first one since we also know the angle at that point in time. However, if we use the second formula we won't need to know *x* as you'll see. Let's differentiate that formula.

$$\sec\theta\tan\theta\ \theta' = \frac{x'}{50}$$

As noted, there are now *x*'s in this formula. We want x' and we know that $\theta = 0.5$ and $\theta' = 0.01$ (do you agree with it being positive?). So, just plug in and solve.

$$\begin{aligned}(50)(0.01)\sec(0.5)\tan(0.5) &= x' \\ x' &= 0.311254 \text{ ft / min}\end{aligned}$$

In this section we've seen three related rates problems. They all work in essentially the same way. The main difference between them was coming up with the relationship between the known and unknown quantities. This is often the hardest part of the problem.

The best way to come up with the relationship is to sketch a diagram that shows the situation. This often seems like a silly step, but can make all the difference in whether we can find the relationship or not.

Higher Order Derivatives

Let's start this section with the following function.

$$f(x) = 5x^3 - 3x^2 + 10x - 5$$

By this point we should be able to differentiate this function. Doing this we get,

$$f'(x) = 15x^2 - 6x + 10$$

Now, this is a function and so it can be differentiated. Here is the notation that we'll use for that as well as the derivative.

$$f''(x) = \left(f'(x)\right)' = 30x - 6$$

This is called the **second derivative** and $f'(x)$ is now called the **first derivative**.

Again, this is a function as so we can differentiate it again. This will be called the **third derivative**. Here is that derivative as well as the notation for the derivative.

$$f'''(x) = \left(f''(x)\right)' = 30$$

Continuing, we can differentiate again. This is called, oddly enough, the **fourth derivative**. We're also going to be changing notation at this point. We can keep adding on primes, but that will get cumbersome after awhile.

$$f^{(4)}(x) = \left(f'''(x)\right)' = 0$$

This process can continue but notice that we will get zero for all derivatives after this point. This set of derivatives leads us to the following fact about the differentiation of polynomials.

Fact

If *p(x)* is a polynomial of degree *n* (*i.e.* the largest exponent in the polynomial) then,

$$p^{(k)}(x) = 0 \qquad \text{for } k \ge n+1$$

We will need to be careful with the "non-prime" notation for derivatives. Consider each of the following.

$$f^{(2)}(x) = f''(x)$$

$$f^{2}(x) = \left[f(x)\right]^2$$

The presence of parenthesis in the exponent denotes differentiation while the absence of parenthesis denotes exponentiation.

Collectively the second, third, fourth, *etc*. derivatives are called **higher order derivatives**.

Let's take a look at some examples of higher order derivatives.

Example 1 Find the first four derivatives for each of the following.

(a) $R(t) = 3t^2 + 8t^{\frac{1}{2}} + \mathbf{e}^t$

(b) $y = \cos x$

(c) $f(y) = \sin(3y) + \mathbf{e}^{-2y} + \ln(7y)$

Solution

(a) There really isn't a lot to do here other than do the derivatives.

$$R'(t) = 6t + 4t^{-\frac{1}{2}} + \mathbf{e}^t$$
$$R''(t) = 6 - 2t^{-\frac{3}{2}} + \mathbf{e}^t$$
$$R'''(t) = 3t^{-\frac{5}{2}} + \mathbf{e}^t$$
$$R^{(4)}(t) = -\frac{15}{2}t^{-\frac{7}{2}} + \mathbf{e}^t$$

Notice that differentiating an exponential function is very simple. It doesn't change with each differentiation.

(b) Again, let's just do some derivatives.

$$\begin{aligned} y &= \cos x \\ y' &= -\sin x \\ y'' &= -\cos x \\ y''' &= \sin x \\ y^{(4)} &= \cos x \end{aligned}$$

Note that cosine (and sine) will repeat every four derivatives.

(c) In the previous two examples we saw some patterns in the differentiation of exponential functions, cosines and sines. We need to be careful however since they only work if there is just a *t* or an *x* in argument. This is the point of this example. In this example we will need to use the chain rule on each derivative.

$$\begin{aligned} f'(y) &= 3\cos(3y) - 2\mathbf{e}^{-2y} + \frac{1}{y} = 3\cos(3y) - 2\mathbf{e}^{-2y} + y^{-1} \\ f''(y) &= -9\sin(3y) + 4\mathbf{e}^{-2y} - y^{-2} \\ f'''(y) &= -27\cos(3y) - 8\mathbf{e}^{-2y} + 2y^{-3} \\ f^{(4)}(y) &= 81\sin(3y) + 16\mathbf{e}^{-2y} - 6y^{-4} \end{aligned}$$

So, we can see with slightly more complicated arguments the patterns that we saw for exponential functions, sines and cosines no longer completely hold.

Let's do a couple more examples.

Example 2 Find the second derivative for each of the following functions.

(a) $Q(x) = \sec(5t)$

(b) $g(w) = \mathbf{e}^{1-2w^3}$

(c) $f(t) = \ln(1+t^2)$

Solution

(a) Here's the first derivative.

$$Q'(x) = 5\sec(5t)\tan(5t)$$

Notice that the second derivative will now require the product rule.

$$\begin{aligned} Q''(x) &= 25\sec(5t)\tan(5t)\tan(5t) + 25\sec(5t)\sec^2(5t) \\ &= 25\sec(5t)\tan^2(5t) + 25\sec^3(5t) \end{aligned}$$

(b) Again, let's start with the first derivative.

$$g'(w) = -6w^2\mathbf{e}^{1-2w^3}$$

As with the first example we will need the product rule for the second derivative.

$$\begin{aligned} g''(w) &= -12w\mathbf{e}^{1-2w^3} - 6w^2\left(-6w^2\right)\mathbf{e}^{1-2w^3} \\ &= -12w\mathbf{e}^{1-2w^3} + 36w^4\mathbf{e}^{1-2w^3} \end{aligned}$$

(c) Same thing here.

$$f'(t) = \frac{2t}{1+t^2}$$

The second derivative this time will require the quotient rule.

$$\begin{aligned} f''(t) &= \frac{2\left(1+t^2\right) - (2t)(2t)}{\left(1+t^2\right)^2} \\ &= \frac{2-2t^2}{\left(1+t^2\right)^2} \end{aligned}$$

As we saw in this last set of examples we will often need to use the product or quotient rule for the higher order derivatives, even when the first derivative didn't require these rules.

Let's work one more example that will illustrate how to use implicit differentiation to find higher order derivatives.

Example 3 Find y'' for

$$x^2 + y^4 = 10$$

Solution

Okay, we know that in order to get the second derivative we need the first derivative and in order to get that we'll need to do implicit differentiation. Here is the work for that.

$$2x + 4y^3y' = 0$$

$$y' = -\frac{x}{2y^3}$$

Now, this is the first derivative. We get the second derivative by differentiating this, which will require implicit differentiation again.

$$\begin{aligned} y'' &= \left(-\frac{x}{2y^3}\right)' \\ &= -\frac{2y^3 - x\left(6y^2 y'\right)}{\left(2y^3\right)^2} \\ &= -\frac{2y^3 - 6xy^2 y'}{4y^6} \\ &= -\frac{y - 3xy'}{2y^4} \end{aligned}$$

This is fine as far as it goes. However, we would like there to be no derivatives in the answer. We don't, generally, mind have x's and/or y's in the answer, but we really don't like derivatives in the answer. We can get rid of the derivative however by acknowledging that we know what the first derivative is and substituting this into the second derivative equation. Doing this gives,

$$\begin{aligned} y'' &= -\frac{y - 3xy'}{2y^4} \\ &= -\frac{y - 3x\left(-\frac{x}{2y^3}\right)}{2y^4} \\ &= -\frac{y + \frac{3}{2}x^2 y^{-3}}{2y^4} \end{aligned}$$

Now that we've found some higher order derivatives we should probably talk about an interpretation of the second derivative.

If the position of an object is given by $s(t)$ we know that the velocity is the first derivative of the position.

$$v(t) = s'(t)$$

The acceleration of the object is the first derivative of the velocity, but since this is the first derivative of the position function we can also think of the acceleration as the second derivative of the position function.

$$a(t) = v'(t) = s''(t)$$

Alternate Notation

There is some alternate notation for higher order derivatives as well. Recall that there was a fractional notation for the first derivative.

$$f'(x) = \frac{df}{dx}$$

We can extend this to higher order derivatives.

$$f''(x) = \frac{d^2y}{dx^2} \qquad\qquad f'''(x) = \frac{d^3y}{dx^3} \qquad\qquad etc.$$

Logarithmic Differentiation

There is one last topic to discuss in this section. Taking the derivatives of some complicated functions can be simplified by using logarithms. This is called **logarithmic differentiation**.

It's easiest to see how this works in an example.

Example 1 Differentiate the function.

$$y = \frac{x^5}{(1-10x)\sqrt{x^2+2}}$$

Solution

Differentiating this function could be done with a product rule and a quotient rule. However, that would be a fairly messy process. We can simplify things somewhat by taking logarithms of both sides.

$$\ln y = \ln\left(\frac{x^5}{(1-10x)\sqrt{x^2+2}}\right)$$

Of course, this isn't really simpler. What we need to do is use the properties of logarithms to expand the right side as follows.

$$\ln y = \ln\left(x^5\right) - \ln\left((1-10x)\sqrt{x^2+2}\right)$$

$$\ln y = \ln\left(x^5\right) - \ln(1-10x) - \ln\left(\sqrt{x^2+2}\right)$$

This doesn't look all the simple. However, the differentiation process will be simpler. What we need to do at this point is differentiate both sides with respect to *x*. Note that this is really implicit differentiation.

$$\frac{y'}{y} = \frac{5x^4}{x^5} - \frac{-10}{1-10x} - \frac{\frac{1}{2}\left(x^2+2\right)^{-\frac{1}{2}}(2x)}{\left(x^2+2\right)^{\frac{1}{2}}}$$

$$\frac{y'}{y} = \frac{5}{x} + \frac{10}{1-10x} - \frac{x}{x^2+2}$$

To finish the problem all that we need to do is multiply both sides by *y* and the plug in for *y* since we do know what that is.

$$\begin{aligned} y' &= y\left(\frac{5}{x}+\frac{10}{1-10x}-\frac{x}{x^2+2}\right) \\ &= \frac{x^5}{(1-10x)\sqrt{x^2+2}}\left(\frac{5}{x}+\frac{10}{1-10x}-\frac{x}{x^2+2}\right) \end{aligned}$$

That was probably a much simpler process of finding the derivative than simply using the quotient and product rule. The answer is probably a lot simpler as well.

We can also use logarithmic differentiation to differentiation functions in the form.

$$y = \left(f(x)\right)^{g(x)}$$

Example 2 Differentiate $y = x^x$

Solution

We've seen two functions similar to this at this point.

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1} \qquad\qquad \frac{d}{dx}\left(a^x\right) = a^x \ln a$$

Neither of these two will work here since both require either the base or the exponent to be a constant. In this case both the base and the exponent are variables and so we have no way to differentiate this function using only known rules from previous sections.

With logarithmic differentiation we can do this however. First take the logarithm of both sides as we did in the first example and use the logarithm properties to simplify things a little.

$$\begin{aligned} \ln y &= \ln x^x \\ \ln y &= x \ln x \end{aligned}$$

Differentiate both sides using implicit differentiation.

$$\frac{y'}{y} = \ln x + x\left(\frac{1}{x}\right) = \ln x + 1$$

As with the first example multiply by *y* and substitute back in for *y*.

$$\begin{aligned} y' &= y\left(1+\ln x\right) \\ &= x^x\left(1+\ln x\right) \end{aligned}$$

Applications of Derivatives

Introduction

In the previous chapter we focused almost exclusively on the computation of derivatives. In this chapter will focus on applications of derivatives. It is important to always remember that we didn't spend a whole chapter talking about computing derivatives just to be talking about them. There are many very important applications to derivatives.

The two main applications that we'll be looking at in this chapter are using derivatives to determine information about graphs of functions and optimization problems. These will not be the only applications however. We will be revisiting limits and taking a look at an application of derivatives that will allow us to compute limits that we haven't been able to compute previously. We will also see how derivatives can be used to estimate solutions to equations.

Here is a listing of the topics in this section.

Critical Points – In this section we will define critical points. Critical points will show up in many of the sections in this chapter so it will be important to understand them.

Minimum and Maximum Values – In this section we will take a look at some of the basic definitions and facts involving minimum and maximum values of functions.

Finding Absolute Extrema – Here is the first application of derivatives that we'll look at in this chapter. We will be determining the largest and smallest value of a function on an interval.

The Shape of a Graph, Part I – We will start looking at the information that the first derivatives can tell us about the graph of a function. We will be looking at increasing/decreasing functions as well as the First Derivative Test.

The Shape of a Graph, Part II – In this section we will look at the information about the graph of a function that the second derivatives can tell us. We will look at inflection points, concavity, and the Second Derivative Test.

The Mean Value Theorem – Here we will take a look that the Mean Value Theorem.

Optimization Problems – This is the second major application of derivatives in this chapter. In this section we will look at optimizing a function, possible subject to some constraint.

L'Hospital's Rule and Indeterminate Forms – This isn't the first time that we've looked at indeterminate forms. In this section we will take a look at L'Hospital's Rule. This rule will allow us to compute some limits that we couldn't do until this section.

Linear Approximations – Here we will use derivatives to compute a linear approximation to a function. As we will see however, we've actually already done this.

Differentials – We will look at differentials in this section as well as an application for them.

Newton's Method – This will be the last application of derivatives that we'll be covering. In this section we'll see how to approximate solutions to an equation.

Critical Points

Critical points will show up through out a majority of this chapter so it makes some sense to start this chapter defining them and working a few examples.

Definition

We say that $x=c$ is a critical point of the function *f(x)* if $f(c)$ exists and if either of the following are true.

$$f'(c)=0 \qquad \text{OR} \qquad f'(c) \text{ doesn't exist}$$

Note that we require that the function to exist at $x=c$ in order for $x=c$ to actually be a critical point. This is an important, and often overlooked, point.

The main point of this section is to work some examples finding critical points. So, let's work some examples.

Example 1 Determine all the critical points for the function.

$$f(x)=6x^5+33x^4-30x^3+100$$

Solution

We first need the derivative of the function in order to find the critical points and so let's get that.

$$\begin{aligned} f'(x) &= 30x^4+132x^3-90x^2 \\ &= 6x^2\left(5x^2+22x-15\right) \\ &= 6x^2(5x-3)(x+5) \end{aligned}$$

Now, our derivative is a polynomial and so will exist everywhere. Therefore the only critical points will be those values of x which make the derivative zero. So, we must solve.

$$6x^2(5x-3)(x+5)=0$$

Form this we can see that there are three critical points. They are,

$$x=-5, \qquad x=0, \qquad x=\frac{3}{5}$$

Polynomials are usually fairly simple function to find critical points for provided the degree doesn't get so large that we have trouble finding the roots of the derivative.

Most of the more "interesting" function for finding critical points aren't polynomials however. So let's take a look at some functions that require a little more effort on our part.

Example 2 Determine all the critical points for the function.

$$g(t) = \sqrt[3]{t^2}\,(2t-1)$$

Solution

To find the derivative it's probably easiest to do a little simplification before we actually differentiate.

$$g(t) = t^{\frac{2}{3}}(2t-1) = 2t^{\frac{5}{3}} - t^{\frac{2}{3}}$$

Now differentiate.

$$g'(t) = \frac{10}{3}t^{\frac{2}{3}} - \frac{2}{3}t^{-\frac{1}{3}} = \frac{10t^{\frac{2}{3}}}{3} - \frac{2}{3t^{\frac{1}{3}}}$$

We will need to be careful with this problem. The derivative of this function doesn't exist at $t=0$ and so that will be one critical point. However, we will also need to determine where the derivative is zero (provided it is of course…).

To help with this it's usually best to combine this into a single rational expression. So, getting a common denominator and combining gives us,

$$g'(t) = \frac{10t-2}{3t^{\frac{1}{3}}}$$

Notice that we still have $t=0$ as a critical point. It's also become much easier to quickly determine where the derivative will be zero. Recall that a rational expression will only be zero if its numerator is zero (and provided the denominator isn't also zero at that point of course).

The numerator will be zero if $t = \frac{1}{5}$ and so there are two critical points for this function.

$$t = 0 \qquad \text{and} \qquad t = \frac{1}{5}$$

Example 3 Determine all the critical points for the function.

$$R(w) = \frac{w^2+1}{w^2-w-6}$$

Solution

Using the quotient rule we get that the derivative is,

$$R'(w) = \frac{-w^2-14w+1}{\left(w^2-w-6\right)^2} = -\frac{w^2+14w-1}{\left(w^2-w-6\right)^2}$$

Notice that we factored a "-1" out of the numerator to help a little with finding the critical points. This negative out in front will not affect the derivative being zero or not existing but will make our work a little easier.

Now, we have two issues to deal with. First the derivative will not exist if there is division by zero in the denominator. So we need to solve,

$$w^2-w-6 = (w-3)(w+2) = 0$$

We didn't bother squaring this since if this is zero, then zero squared is still zero and if it isn't zero then squaring it won't make it zero.

So, we can see from this that the derivative will not exist at $w=3$ and $w=-2$. However, these are NOT critical points since the function will also not exist at these points. Recall that in order for a point to be a critical point the function must actually exist at that point.

At this point we need to be careful. The numerator doesn't factor, but that doesn't mean that there aren't any critical points where the derivative is zero. We can use the quadratic formula on the numerator to determine if the fraction as a whole is ever zero.

$$w = \frac{-14 \pm \sqrt{(14)^2 - 4(1)(-1)}}{2(1)} = \frac{-14 \pm \sqrt{200}}{2} = \frac{-14+10\sqrt{2}}{2} = -7 \pm 5\sqrt{2}$$

So, we get two more critical points. Also, these are not "nice" integers or fractions. This will happen on occasion. Don't get too locked into answers always being "nice". Often they aren't.

Note as well that we only use real numbers for critical points. So, if upon solving the quadratic in the numerator, we had gotten complex number these would not have been considered critical points.

Summarizing, we have two critical points. They are,

$$w = -7+5\sqrt{2}, \quad w = -7-5\sqrt{2}$$

So far all the examples have not had any trig functions, exponential functions, *etc*. in them. We shouldn't expect that to always be the case. So, let's take a look at some examples that don't just involve powers of x.

Example 4 Determine all the critical points for the function.

$$y = 6x - 4\cos(3x)$$

Solution

First get the derivative and don't forget to use the chain rule on the second term.

$$y' = 6 + 12\sin(3x)$$

Now, this will exist everywhere and so there won't be any critical points for which the derivative doesn't exist. The only critical points will come from points that make the derivative zero. We will need to solve,

$$6 + 12\sin(3x) = 0$$

$$\sin(3x) = -\frac{1}{2}$$

From the unit circle we have the following solutions.

$$3x = \frac{7\pi}{6} + 2\pi n, \qquad n = 0, \pm 1, \pm 2, \ldots$$

$$3x = \frac{11\pi}{6} + 2\pi n, \qquad n = 0, \pm 1, \pm 2, \ldots$$

Don't forget the $2\pi n$ on these! There will be problems down the road in which we will miss solutions without this! Also make sure that it gets put on at this stage! Now divide by 3 to get all the critical points for this function.

$$x = \frac{7\pi}{18} + \frac{2\pi n}{3}, \qquad n = 0, \pm 1, \pm 2, \ldots$$

$$x = \frac{11\pi}{18} + \frac{2\pi n}{3}, \qquad n = 0, \pm 1, \pm 2, \ldots$$

Notice that in the previous example we got an infinite number of critical points. That will happen on occasion.

Example 5 Determine all the critical points for the function.

$$h(t) = 10t\mathbf{e}^{3-t^2}$$

Solution

Here's the derivative for this function.

$$h'(t) = 10\mathbf{e}^{3-t^2} + 10t\mathbf{e}^{3-t^2}(-2t) = 10\mathbf{e}^{3-t^2} - 20t^2\mathbf{e}^{3-t^2}$$

Now, this looks unpleasant, however with a little factoring we can clean things up a little.

$$h'(t) = 10\mathbf{e}^{3-t^2}\left(1 - 2t^2\right)$$

This function will exist everywhere and so no critical points will come from that.

Determining where this is zero is easier than it looks. We know that exponentials are never zero and so the only way the derivative will be zero is if,

$$\begin{aligned} 1-2t^2 &= 0 \\ 1 &= 2t^2 \\ \frac{1}{2} &= t^2 \end{aligned}$$

We will have two critical points for this function.

$$t = \pm\frac{1}{\sqrt{2}}$$

Example 6 Determine all the critical points for the function.

$$f(x) = x^2 \ln(3x) + 6$$

Solution

Before getting the derivative let's notice that since we can't take the log of a negative number or zero we will only be able to look at $x>0$.

The derivative is then,

$$\begin{aligned} f'(x) &= 2x\ln(3x) + x^2\left(\frac{3}{3x}\right) \\ &= 2x\ln(3x) + x \\ &= x\left(2\ln(3x)+1\right) \end{aligned}$$

Now, this derivative will not exist if x is a negative number or if $x=0$, but then again neither will the function and so these are not critical points. If $x>0$ the function will exist and so the only thing we need to worry about is where the derivative is zero.

First note that despite appearances the derivative will not be zero for $x=0$. For $x=0$ the derivative doesn't exist because of the natural logarithm and so the derivative can't be zero there!

So, the derivative will only be zero if,

$$\begin{aligned} 2\ln(3x)+1 &= 0 \\ \ln(3x) &= -\frac{1}{2} \end{aligned}$$

Recall that we can solve this by exponentiating both sides.

$$\mathbf{e}^{\ln(3x)} = \mathbf{e}^{-\frac{1}{2}}$$
$$3x = \mathbf{e}^{-\frac{1}{2}}$$
$$x = \frac{1}{3}\mathbf{e}^{-\frac{1}{2}} = \frac{1}{3\sqrt{\mathbf{e}}}$$

This is the single critical point for this function.

Let's work one more problem to make a point.

Example 7 Determine all the critical points for the function.

$$f(x) = x\mathbf{e}^{x^2}$$

Solution

Note that this function is not much different from the function used in Example 5. In this case the derivative is,

$$f'(x) = \mathbf{e}^{x^2} + x\mathbf{e}^{x^2}(2x) = \mathbf{e}^{x^2}(1+2x^2)$$

This function will never be zero for any real value of x. The exponential is never zero of course and the polynomial will only be zero if x is complex and recall that we only want real values of x for critical points.

Therefore, this function will not have any critical points.

It is important to note that not all functions will have critical points! In this course most of the functions that we will be looking at do have critical points. That is only because those problems make for more interesting examples. Do not let this fact lead you to always expect that a function will have critical points. Sometimes they don't as this final example has shown.

Minimum and Maximum Values

Many of our applications in this chapter will revolve around minimum and maximum values of a function. While we can all visualize the minimum and maximum values of a function we want to be a little more specific in our work here. In particular we want to differentiate between two types of minimum or maximum values. The following definition gives the types of minimums and/or maximums values that we'll be looking at.

Definition

1. We say that $f(x)$ has an **absolute (or global) maximum** at $x=c$ if $f(x) \le f(c)$ for every x in the domain we are working on.
2. We say that $f(x)$ has a **relative (or local) maximum** at $x=c$ if $f(x) \le f(c)$ for every

x in some open interval around $x=c$.

3. We say that $f(x)$ has an **absolute (or global) minimum** at $x=c$ if $f(x) \geq f(c)$ for every x in the domain we are working on.
4. We say that $f(x)$ has a **relative (or local) minimum** at $x=c$ if $f(x) \geq f(c)$ for every x in some open interval around $x=c$.

Also, we call the minimum and maximum points of a function the **extrema** of the function. So, relative extrema will refer to the relative minimums and maximums while absolute extrema refer to the absolute minimums and maximums.

Now, let's talk a little bit about the subtle difference between the absolute and relative in the definition above.

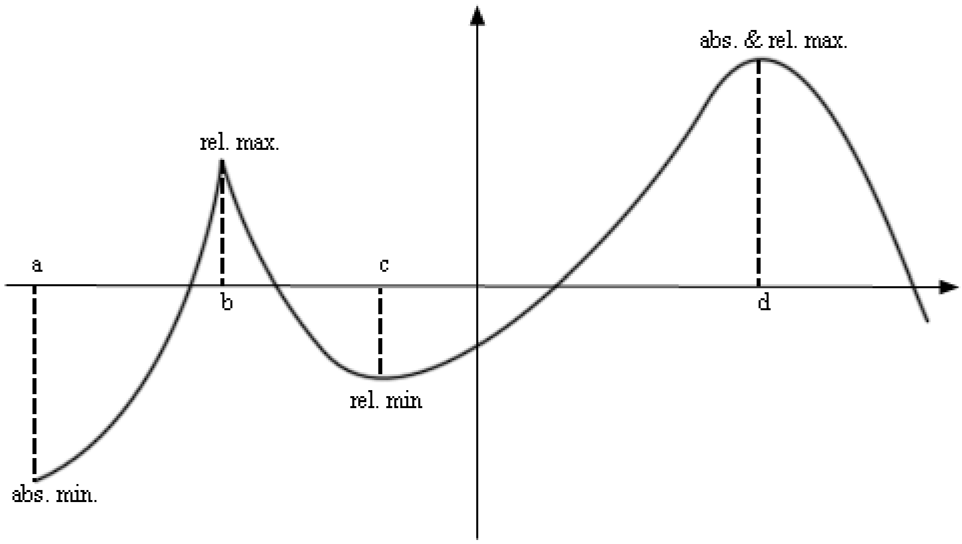
We will have an absolute maximum or minimum at $x=c$ provided $f(c)$ is the largest or smallest value that the function will ever take on the domain that we are working on. Also, when we say the "domain we are working on" this simply means the range of x's that we have chosen to work with for a given problem. There may be other values of x that we can actually plug into the function but have excluded them for some reason.

A relative maximum or minimum is slightly different. All that's required for a point to be a relative maximum or minimum is for that point to be a maximum or minimum in some interval of x's around $x=c$. There may be larger or smaller values of the function at some other place, but relative to $x=c$, or local to $x=c$, $f(c)$ is larger or smaller than all the other function values that are near it.

Note as well that in order for a point to be a relative extrema we must be able to look at function values on both sides of $x=c$ to see if it really is a maximum or minimum at that point. This means that relative extrema do not occur at the end points of a domain. They can only occur interior to the domain.

There is actually some debate on the preceding point. Some folks do feel that relative extrema can occur on the end points of a domain. However, in this class we will be using the definition that says that they can't occur at the end points of a domain.

It's usually easier to get a feel for the definitions by taking a quick look at a graph.



For the function shown in this graph we have relative maximums at $x=b$ and $x=d$. Both of these point are relative maximums since they are interior to the domain shown and are the largest point on the graph in some interval around the point. The point $x=a$ is an absolute minimum since it is the lowest point on the graph. We also have a relative minimum at $x=c$ since this point is interior to the domain and is the lowest point on the graph in an interval around it. The far right end point will not be a relative minimum since it is an end point.

The function will have an absolute maximum at $x=d$ and an absolute minimum at $x=a$. These two points are the largest and smallest that the function will ever be. We can also notice that the absolute extrema for a function will occur at either the endpoints of the domain or at relative extrema. We will use this idea in later sections so it's more important than it might seem at the present time.

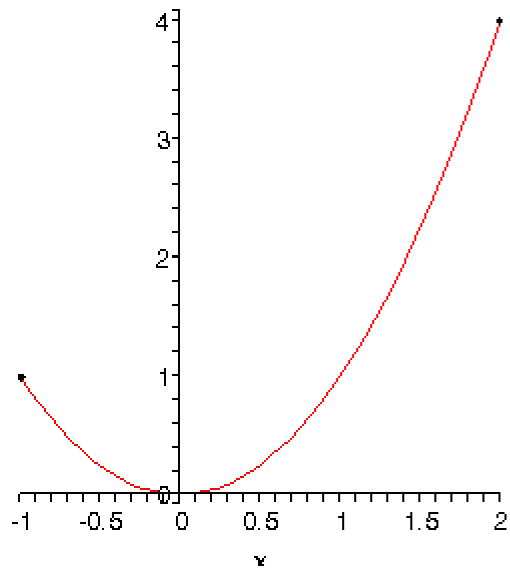
Let's take a quick look at some examples to make sure that we have the definitions of absolute extrema and relative extrema straight.

Example 1 Identify the absolute extrema and relative extrema for the following function.

$$f(x) = x^2 \qquad \text{on} \qquad [-1,2]$$

Solution

Since this function is easy enough to graph let's do that. However, we only want the graph on the interval [-1,2]. Here is the graph,



Note that we used dots at the end of the graph to remind us that the graph ends at these points.

We can now identify the extrema from the graph. It looks like we've got a relative and absolute minimum of zero at $x=0$ and an absolute maximum of four at $x=2$. Note that $x=-1$ is not a relative maximum since it is at the end point of the interval.

This function doesn't have any relative maximums.

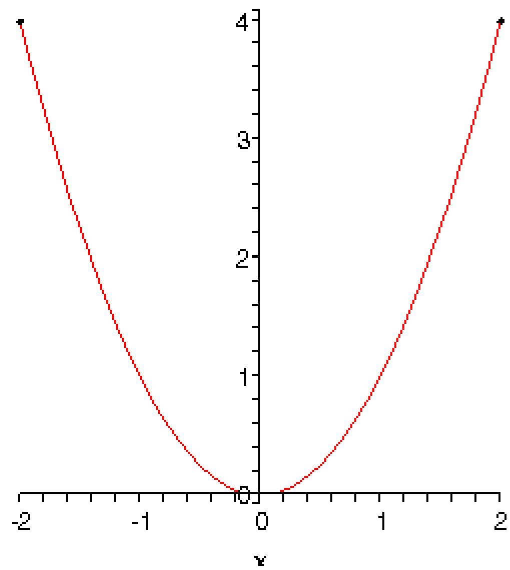
As we saw in the previous example functions do not have to have relative extrema.

Example 2 Identify the absolute extrema and relative extrema for the following function.

$$f(x) = x^2 \qquad \text{on} \qquad [-2,2]$$

Solution

Here is the graph for this function.



In this case we still have a relative and absolute minimum of zero at $x=0$. We also still have an absolute maximum of four. However, unlike the first example this will occur at two points, $x=-2$ and $x=2$.

Again, the function doesn't have any relative maximums.

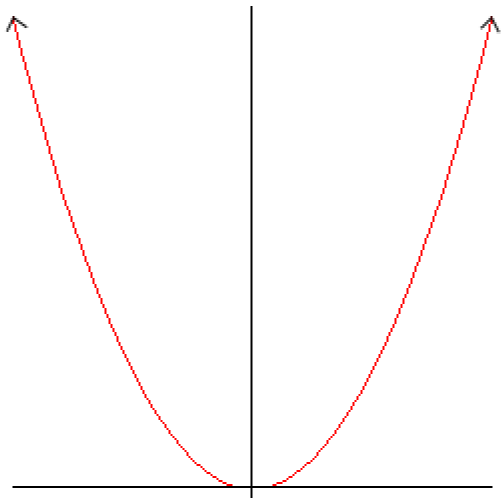
As this example has shown there can only be a single absolute maximum or absolute minimum value, but it can occur at more than one place in the domain.

Example 3 Identify the absolute extrema and relative extrema for the following function.

$$f(x) = x^2$$

Solution

In this case we've given no domain and so the assumption is that we will take the largest possible domain. For this function that means all the real numbers. Here is the graph.



In this case the graph doesn't stop increasing at either end and so there are no maximums of any kind for this function. No matter which point we pick on the graph there will be points both larger and smaller than it on either side so we can't have any maximums in a graph.

We still have a relative and absolute minimum value of zero at $x=0$.

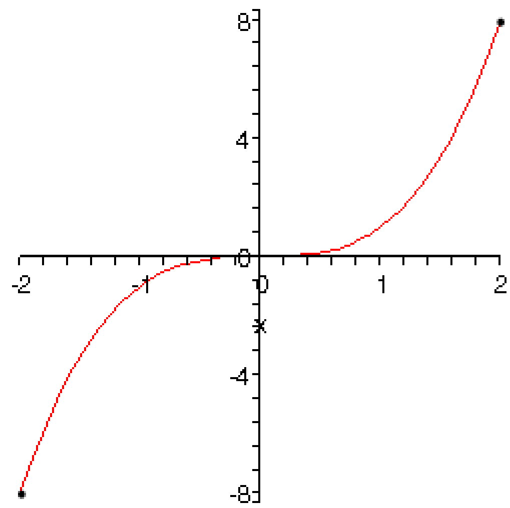
So, some graphs can have minimums but not maximums. Likewise, a graph could have maximums but not minimums.

Example 4 Identify the absolute extrema and relative extrema for the following function.

$$f\left(x\right)=x^3 \qquad \text{on} \qquad \left[-2,2\right]$$

Solution

Here is the graph for this function.



This function has an absolute maximum of eight at $x=2$ and an absolute minimum of negative eight at $x=-2$.

This function has no relative extrema.

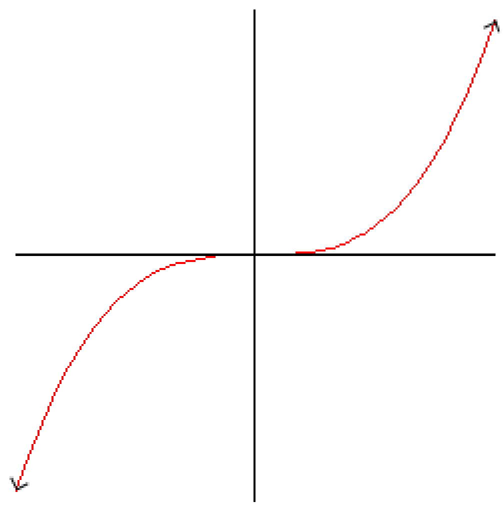
So, a function doesn't have to relative extrema as this example has shown.

Example 5 Identify the absolute extrema and relative extrema for the following function.

$$f(x) = x^3$$

Solution

Again, we aren't restricting the domain this time so here's the graph.



In this case the function has no relative extrema and no absolute extrema.

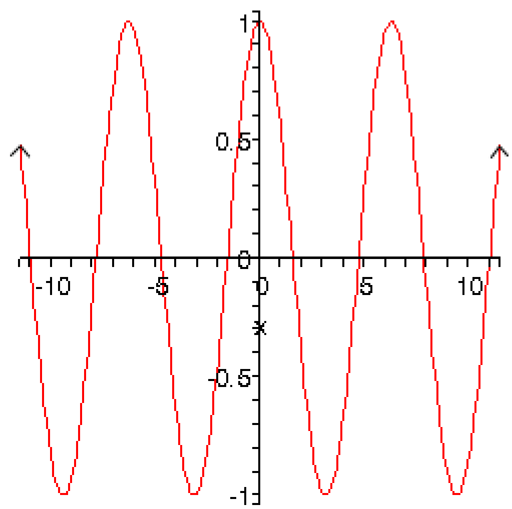
This example shows us that functions don't have to have any kind of extrema.

Example 6 Identify the absolute extrema and relative extrema for the following function.

$$f(x) = \cos(x)$$

Solution

We've not restricted the domain for this function. Here is the graph.



Cosine has many extrema. Cosine will have both relative and absolute maximums of 1 at

$$x = \ldots -4\pi, -2\pi, 0, 2\pi, 4\pi, \ldots$$

Cosine will have both relative and absolute minimums of -1 at

$$x = \ldots -3\pi, -\pi, 0, \pi, 3\pi, \ldots$$

We've now worked quite a few examples and we can use these examples to see a nice fact about absolute extrema. First let's notice that all the functions above were continuous functions. Next notice that every time we restricted the domain to a closed interval (*i.e.* the interval contains its end points) we got absolute maximums and absolute minimums. Finally, in only one of the three examples in which we did not restrict the domain did we get both an absolute maximum and an absolute minimum.

These observations lead us the following theorem.

Extreme Value Theorem

Suppose that *f(x)* is continuous on the interval [*a*,*b*] then there are two numbers $a \le c, d \le b$ so that *f(c)* is an absolute maximum for the function and *f(d)* is an absolute minimum for the function.

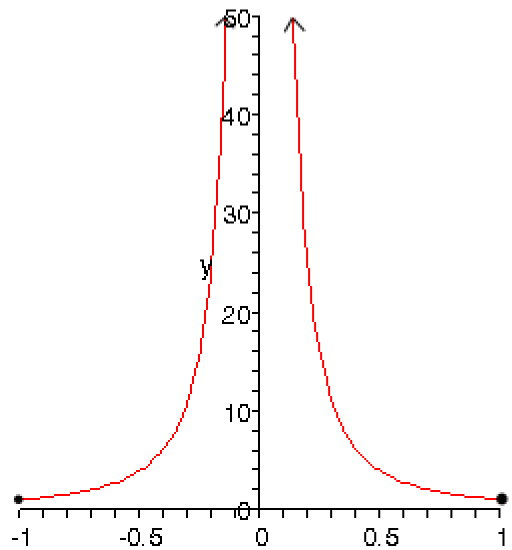
So, if we have a continuous function on an interval [*a*,*b*] then we are guaranteed to have both an absolute maximum and an absolute minimum for the function somewhere in the interval. The theorem doesn't tell us where they will occur or if they will occur more than once, but at least it tells us that they do exist somewhere. Sometimes, all that we need to know is that they do exist.

This theorem doesn't say anything about absolute extrema if we aren't working on an interval. We saw examples of functions above that had both absolute extrema, one absolute extrema, and no absolute extrema when we didn't restrict ourselves down to an interval.

The requirement that a function be continuous is also required in order for us to use the theorem. Consider the case of

$$f(x) = \frac{1}{x^2} \qquad \text{on} \qquad [-1,1]$$

Here's the graph.



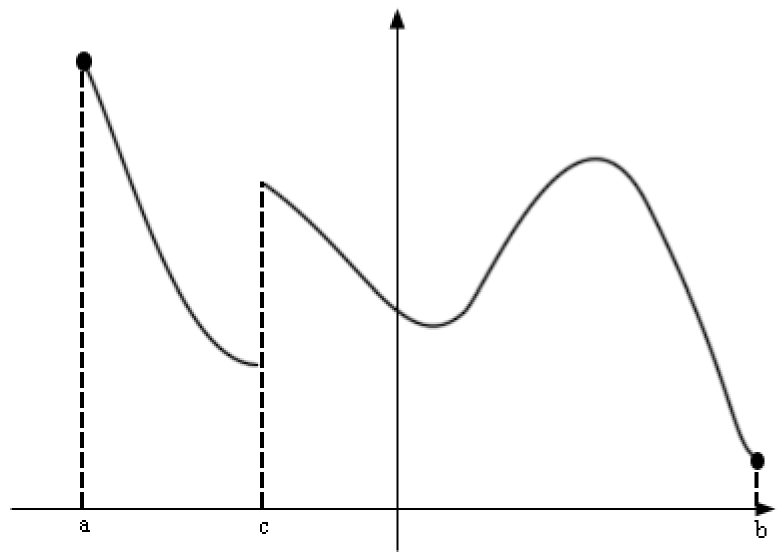
This function is not continuous at *x=0* as we move in towards zero the function approaching infinity. So, the function does not have an absolute maximum. Note that it does have an absolute minimum however. In fact the absolute minimum occurs twice at both *x=-1* and *x=1*.

If we changed the interval a little to say,

$$f(x) = \frac{1}{x^2} \qquad \text{on} \qquad \left[\frac{1}{2},1\right]$$

the function would now have both absolute extrema. We may only run into problems if the interval contains the point of discontinuity. If it doesn't then the theorem will hold.

We should also point out that just because a function is not continuous at a point that doesn't mean that it won't have both absolute extrema in an interval that contains that point. Below is the graph of a function that is not continuous at a point in the given interval and yet has both absolute extrema.



This graph is not continuous at $x=c$, yet it does have both an absolute maximum ($x=a$) and an absolute minimum ($x=b$).

The point of all this is that we need to be careful to only use the Extreme Value Theorem when the conditions of the theorem are met and not misinterpret the results if the conditions aren't met.

In order to use the Extreme Value Theorem we must have an interval and the function must be continuous on that interval. If we don't have an interval and/or the function isn't continuous on the interval then the function may or may not have absolute extrema.

We need to discuss one final topic in this section before moving on to the first major application of the derivative.

Fermat's Theorem

If $f(x)$ has a relative extrema at $x=c$ then $x=c$ is a critical point of $f(x)$.

This theorem tells us that there is a nice relationship between relative extrema and critical points. In fact it will allow us to get a list of all possible relative extrema. Since a relative extrema must be a critical point the list of all critical points will give us a list of all possible relative extrema.

Consider the case of $f(x)=x^2$. We saw that this function had a relative minimum at $x=0$. So according to Fermat's theorem $x=0$ should be a critical point. The derivative of the function is,

$$f'(x) = 2x$$

Sure enough $x=0$ is a critical point.

Be careful not to misuse this theorem. It doesn't say that a critical point will be a relative extrema. To see this, consider the following case.

$$f(x) = x^3 \qquad\qquad f'(x) = 3x^2$$

Clearly $x=0$ is a critical point. However we saw earlier that this function has no relative extrema of any kind. So, critical points do not have to be relative extrema.

Also note that this theorem says nothing about absolute extrema. An absolute extrema may or may not be a critical point.

As we will see this theorem will be key to several of our applications in this chapter.

Finding Absolute Extrema

It's now time to see our first major application of derivatives in this chapter. Given a continuous function, $f(x)$, on an interval $[a,b]$ we want to determine the absolute extrema of the function. To do this we will need many of the ideas that we looked at in the previous section.

First, since we have an interval and we are assuming that the function is continuous the Extreme Value Theorem tells us that we can in fact do this. This is a good thing of course. We don't want to be trying to find something that doesn't exist.

Next, we saw in the previous section that absolute extrema will occur at endpoints or relative extrema. Also, from Fermat's Theorem we know that the list of critical points is also a list of all possible relative extrema. So the endpoints along with the list of all critical points will in fact be a list of all possible absolute extrema.

Now we just need to recall that the absolute extrema are nothing more than the largest and smallest values that a function will take so all that we really need to do is get a list of possible absolute extrema, plug these points into our function and then identify the largest and smallest values.

Here is the procedure for finding absolute extrema.

Finding Absolute Extrema of $f(x)$ on $[a,b]$.

1. Find all critical points of $f(x)$ that are in the interval $[a,b]$. This makes sense if you think about it. Since we are only interested in what the function is doing in this interval we don't care about critical points that fall outside the interval.
2. Evaluate the function at the critical points found in step 1 and the end points.
3. Identify the absolute extrema.

There really isn't a whole lot to this procedure. The first step in this process will be the most labor intensive. The remaining two steps are not terribly difficult steps or time consuming.

Let's do some examples.

Example 1 Determine the absolute extrema for the following function and interval.

$$g(t) = 2t^3 + 3t^2 - 12t + 4 \qquad \text{on} \qquad [-4, 2]$$

Solution

All we really need to do here is follow the procedure given above. So, the first step is to get the derivative so that we can find the critical points of the function.

$$g'(t) = 6t^2 + 6t - 12 = 6(t+2)(t-1)$$

It looks like we'll have two critical points, $t=-2$ and $t=1$. Note that we actually want something more than just the critical points. We only want the critical points of the function that lie in the interval in question. Both of these do fall in the interval as so we will use both of them. That may seem like a silly thing to mention at this point, but it is often forgotten and so we will mention it at every opportunity to make it's not forgotten.

Now we evaluate the function at the critical points and the end points of the interval.

$$g(-2) = 24 \qquad\qquad g(1) = -3$$
$$g(-4) = -28 \qquad\qquad g(2) = 8$$

Absolute extrema are the largest and smallest the function will ever be and these four points represent the only places in the interval where the absolute extrema can occur. So, from this list we see that the absolute maximum of $g(t)$ is 24 and it occurs at $t=-2$ (a critical point) and the absolute minimum of $g(t)$ is -28 which occurs at $t=-4$ (an endpoint).

In this example we saw that absolute extrema can and will occur at both endpoints and critical points. One of the biggest mistakes that students make with these problems is to forget to check the endpoints of the interval.

Example 2 Determine the absolute extrema for the following function and interval.

$$g(t) = 2t^3 + 3t^2 - 12t + 4 \qquad \text{on} \qquad [0, 2]$$

Solution

Note that this problem is almost identical to the first problem. The only difference is the interval that we're working on. This small change will completely change our answer. With this change we have excluded both of the answers from the first example.

The first step is to again find the critical points. From the first example we know these are $t=-2$ and $t=1$. At this point it's important to recall that we only want the critical points that actually fall in the interval in question. This means that we only want $t=1$ since $t=-2$ falls outside the interval.

Now evaluate the function at the single critical point in the interval and the two endpoints.

$$g(1) = -3 \qquad g(0) = 4 \qquad g(2) = 8$$

From this list of values we see that the absolute maximum is 8 and will occur at *t=2* and the absolute minimum is -3 which occurs at *t=1*.

As we saw in this example a simple change in the interval can completely change the answer. It also has shown us that we do need to be careful to exclude critical points that aren't in the interval. Had we forgotten this and included *t=-2* we would have gotten the wrong absolute maximum!

This is the other big mistakes that students make in these problems. All too often they forget to exclude critical points that aren't in the interval. If your instructor is anything like me this will mean that you will get the wrong answer. It's not to hard to make sure that a critical point outside of the interval is larger or smaller than any of the points in the interval.

Example 3 Suppose that the population (in thousands) of a certain kind of insect after *t* months is given by the following formula.

$$P(t) = 2\sqrt{3}\,t + \sin(4t) + 100$$

Determine the minimum and maximum population in the first 4 months.

Solution

The question that we're really asking is to find the absolute extrema of *P(t)* on the interval [0,4]. We know how to do this.

Let's start with the derivative.

$$P'(t) = 2\sqrt{3} + 4\cos(4t)$$

We need the critical points of the function. The derivative exists everywhere so there are no critical points from that. So, all we need to do is determine where the derivative is zero.

$$2\sqrt{3} + 4\cos(4t) = 0$$

$$\cos(4t) = -\frac{\sqrt{3}}{2}$$

The solutions to this are,

$$\begin{aligned} 4t &= \frac{5\pi}{6} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots \\ 4t &= \frac{7\pi}{6} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots \end{aligned} \quad \Rightarrow \quad \begin{aligned} t &= \frac{5\pi}{24} + \frac{\pi n}{2}, \quad n = 0, \pm 1, \pm 2, \ldots \\ t &= \frac{7\pi}{24} + \frac{\pi n}{2}, \quad n = 0, \pm 1, \pm 2, \ldots \end{aligned}$$

So, these are all the critical points. We need to determine the ones that fall in the interval [0,4]. There's nothing to do except plug some *n*'s into the formulas until we get all of them.

n=0 **:**

$$t = \frac{5\pi}{24} = 0.654498$$
$$t = \frac{7\pi}{24} = 0.916298$$

We'll need both of these critical points.

n=1 **:**

$$t = \frac{5\pi}{24} + \frac{\pi}{2} = \frac{17\pi}{24} = 2.225295$$
$$t = \frac{7\pi}{24} + \frac{\pi}{2} = \frac{19\pi}{24} = 2.487094$$

We'll need these.

n=2 **:**

$$t = \frac{5\pi}{24} + \pi = \frac{29\pi}{24} = 3.796091$$
$$t = \frac{7\pi}{24} + \pi = \frac{31\pi}{24} = 4.057891$$

In this case we only need the first one since the second is out of the interval.

There are five critical points that are in the interval. They are,

$$\frac{5\pi}{24}, \frac{7\pi}{24}, \frac{17\pi}{24}, \frac{19\pi}{24}, \frac{29\pi}{24}$$

Finally, to determine the absolute minimum and maximum population we only need to plug these values into the function as well as the two end points. Here are the function evaluations.

$$P(0) = 100.0 \qquad P(4) = 8\sqrt{3} + \sin(16) + 100 = 113.569$$
$$P\left(\frac{5\pi}{24}\right) = \frac{5\sqrt{3}\,\pi}{12} + \frac{201}{2} = 102.767 \qquad P\left(\frac{7\pi}{24}\right) = \frac{7\sqrt{3}\,\pi}{12} + \frac{199}{2} = 102.674$$
$$P\left(\frac{17\pi}{24}\right) = \frac{17\sqrt{3}\,\pi}{12} + \frac{201}{2} = 108.209 \qquad P\left(\frac{19\pi}{24}\right) = \frac{19\sqrt{3}\,\pi}{12} + \frac{199}{2} = 108.116$$
$$P\left(\frac{29\pi}{24}\right) = \frac{19\sqrt{3}\,\pi}{12} + \frac{201}{2} = 113.650$$

From these evaluations it appears that the minimum population is 100,000 (remember that *P* is in thousands…) which occurs at *t=0* and the maximum population is 113,650 which occurs at $t = \frac{29\pi}{24}$.

Make sure that you can correctly solve trig equations. If we had forgotten the $2\pi n$ we would have missed the last three critical points in the interval and hence gotten the wrong answer since the maximum population was at the final critical point.

Example 4 Suppose that the amount of money in a bank account after *t* years is given by,

$$A(t) = 2000 - 10t\mathbf{e}^{5-\frac{t^2}{8}}$$

Determine the minimum and maximum amount of money in the account during the first 10 years that it is open.

Solution

Here we are really asking for the absolute extrema of *A(t)* on the interval [0,10].

We'll first need the derivative so we can find the critical points.

$$\begin{aligned} A'(t) &= -10\mathbf{e}^{5-\frac{t^2}{8}} - 10t\mathbf{e}^{5-\frac{t^2}{8}}\left(-\frac{t}{4}\right) \\ &= 10\mathbf{e}^{5-\frac{t^2}{8}}\left(-1+\frac{t^2}{4}\right) \end{aligned}$$

The derivative exists everywhere and the exponential is never zero. Therefore the derivative will only be zero where,

$$-1+\frac{t^2}{4} = 0 \qquad \Rightarrow \qquad t^2 = 4 \qquad \Rightarrow \qquad t = \pm 2$$

We've got two critical points, however only *t=2* is actually in the interval so that is only critical point that we'll use.

Let's now evaluate the function at the lone critical point and the end points of the interval. Here are those function evaluations.

$$A(0) = 2000 \qquad A(2) = 199.66 \qquad A(10) = 1999.94$$

So, the maximum amount in the account will be \$2000 which occurs at *t=0* and the minimum amount in the account will be \$199.66 which occurs at the 2 year mark.

In this example there are two important things to note. First, if we had included the second critical point we would have gotten an incorrect answer for the maximum amount so it's important to be careful with which critical points to include and which to exclude.

Also, don't get too carried away with rounding! If we round too much we would have gotten the maximum occurring twice when in reality it doesn't really.

All of the problems that we've worked to this point had derivatives that existed everywhere and so the only critical points that we looked at where those for which the derivative is zero. Do not get too locked into this always happening. Most of the problems that we run into will be like this, but they won't all be like this.

Let's work another example to make this point.

Example 5 Determine the absolute extrema for the following function and interval.

$$Q(y) = 3y(y+4)^{\frac{2}{3}} \qquad \text{on} \qquad [-5,-1]$$

Solution

First we'll need the derivative.

$$\begin{aligned} Q'(y) &= 3(y+4)^{\frac{2}{3}} + 3y\left(\frac{2}{3}\right)(y+4)^{-\frac{1}{3}} \\ &= 3(y+4)^{\frac{2}{3}} + \frac{2y}{(y+4)^{\frac{1}{3}}} \\ &= \frac{3(y+4)+2y}{(y+4)^{\frac{1}{3}}} \\ &= \frac{5y+12}{(y+4)^{\frac{1}{3}}} \end{aligned}$$

So, it looks like we've got two critical points.

$$\begin{aligned} y &= -4 && \text{Because the derivative doesn't exist here.} \\ y &= -\frac{12}{5} && \text{Because the derivative is zero here.} \end{aligned}$$

Both of these are in the interval so let's evaluate the function at these points and the end points of the interval.

$$\begin{aligned} Q(-4) &= 0 & \qquad Q\left(-\frac{12}{5}\right) &= -9.849 \\ Q(-5) &= -15 & \qquad Q(-1) &= -6.241 \end{aligned}$$

The function has an absolute maximum of zero at *y=-4* and the function will have an absolute minimum of -15 at *y=-5*.

So, if we had ignored or forgotten about the critical point at *y=-4* we would not have gotten the correct answer.

In this section we've seen how we can use a derivative to identify the absolute minimums and maximums of a function. This is an important application of derivatives that will arise from time to time so don't forget about it.

The Shape of a Graph, Part I

In the previous section we saw how to use the derivative to determine the absolute minimum and maximum values of a function. However, there is a lot more information about a graph that can be determined from the first derivative of a function. We will start looking at that information in this section.

By the time this section is over we will be able to identify all the relative extrema of a function. We will also revisit a familiar interpretation of the derivative.

In fact, let's start with that interpretation. Let's suppose that we have a function, *f(x)*. We know that one of the interpretations of the first derivative, $f'(x)$, is the rate of change of the function. Back when we were looking at rates of change in the previous chapter we used the fact that if the rate of change was a positive number then the function was increasing and if the rate of change was a negative number then the function was decreasing. We also used the fact that if the rate of change of the function was zero then the function was not changing.

Since rates of change of the function are nothing more than the derivative we can summarize these observations up with the following fact.

Fact

1. If $f'(x) > 0$ for every *x* on some interval *I*, then *f(x)* is increasing on the interval.
2. If $f'(x) < 0$ for every *x* on some interval *I*, then *f(x)* is decreasing on the interval.
3. If $f'(x) = 0$ for every *x* on some interval *I*, then *f(x)* is constant on the interval.

Let's take a look at an example.

Example 1 Determine all intervals where the following function is increasing or decreasing.

$$f(x) = -x^5 + \frac{5}{2}x^4 + \frac{40}{3}x^3 + 5$$

Solution

To determine if the function is increasing or decreasing we will need the derivative.

$$\begin{aligned} f'(x) &= -5x^4 + 10x^3 + 40x^2 \\ &= -5x^2\left(x^2 - 2x - 8\right) \\ &= -5x^2\left(x-4\right)\left(x+2\right) \end{aligned}$$

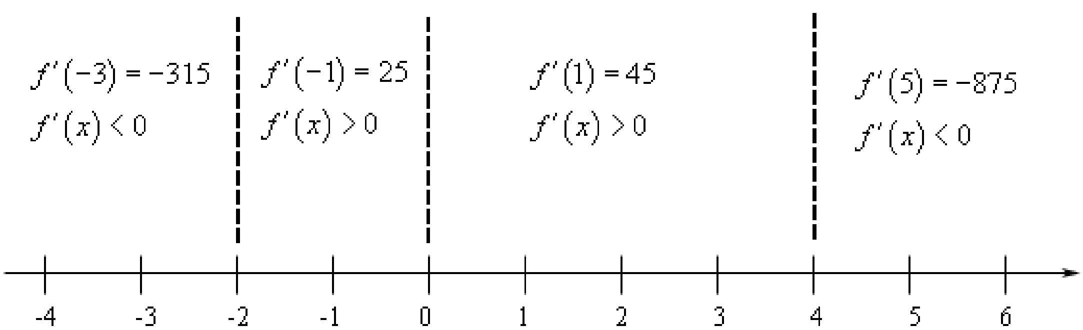
Note that when we factored the derivative we first factored a "-1" out to make the rest of the factoring a little easier.

From the factored form of the derivative we see that we have three critical points for this function : $x=-2$, $x=0$, and $x=4$. We'll need these in a bit.

Now, we need to determine where the derivative is positive and where it's negative. We've done this a couple of times now in both the Review chapter and the previous chapter. Since the derivative is a polynomial it is continuous and so we know that the only way for it to change signs is to first go through zero.

In other words, the only place that the derivative *may* change signs is at the critical points of the function. We've now got another use for critical points. So, we'll build a number line, graph the critical points and pick test points from each region to see if the derivative is positive or negative in each region.

Here is the number line and the test points.



Make sure that you test your points in the derivative. One of the more common mistakes here is to test the points in the function instead!

Recall that we know that the derivative will be the same sign in each region. The only place that the derivative can change signs is at the critical points and we've marked the only critical points on the number line.

So, it looks we've got the following intervals of increase and decrease.

$$\begin{aligned} &\text{Increase :} \quad -2 < x < 0 \text{ and } 0 < x < 4 \\ &\text{Decrease :} \ -\infty < x < -2 \text{ and } 4 < x < \infty \end{aligned}$$

Note that often the fact that only a single point separates the two intervals of increase will be ignored and the interval will be written *-2 < x < 4*.

In this example we used the fact that the only place that a derivative can change sign is at the critical points. Also, the critical points for this function were those for which the derivative was zero. However, the same thing can be said for critical points where the derivative doesn't exist. This is nice to know. A function can change signs where it is zero or doesn't exist.

A nice consequence of knowing where a function increases and where a function decreases is that we can know start to sketch a graph of the function.

Example 2 Sketch the graph of the following function.

$$f(x) = -x^5 + \frac{5}{2}x^4 + \frac{40}{3}x^3 + 5$$

Solution

There really isn't a whole lot to this example. Whenever we sketch a graph it's nice to have a few points on the graph to give us a starting place. So we'll start by graphing the value of the function at the critical points. These points are,

$$f(-2) = -\frac{89}{3} = -29.67 \qquad f(0) = 5 \qquad f(4) = \frac{1423}{3} = 474.33$$

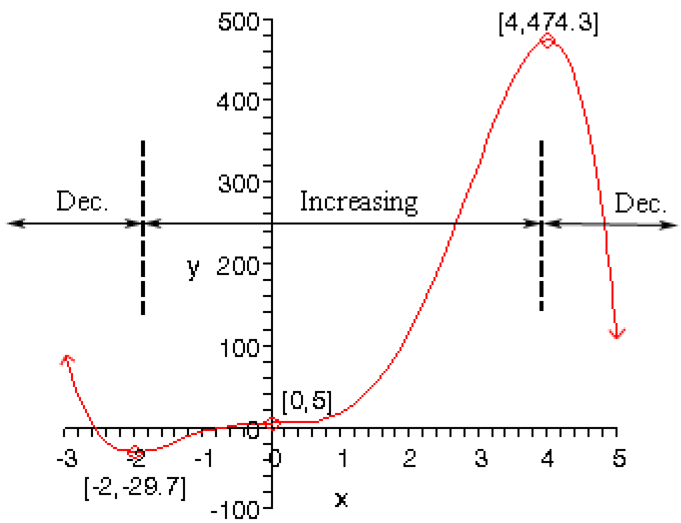
Once these points are graphed we go to the increasing and decreasing information and start sketching. For reference purposes here is the increasing/decreasing information.

$$\text{Increase : } \quad -2 < x < 0 \text{ and } 0 < x < 4$$

$$\text{Decrease : } -\infty < x < -2 \text{ and } 4 < x < \infty$$

Note that we are only after a sketch of the graph. We won't be able to accurately predict the curvature of the graph at this point. That's actually a topic for the next section. However, even without this information we will still be able to get a basic idea of what the graph should look like.

Here is the graph of the function.



Note that we also know that the graph will be horizontal when it goes through each of the critical points. For each of the critical points the derivative was zero and so we know that the tangent line for each of these points must be horizontal.

Recall Fermat's Theorem from the Minimum and Maximum Values section. This theorem told us that all relative extrema of a function will be critical points. The graph in the previous example has two relative extrema and both occur at critical points. Note as well that we've got a critical point that isn't a relative extrema ($x=0$). This is okay since Fermat's theorem doesn't say that all critical points will be relative extrema. It only states that relative extrema will be critical points.

The graph in the previous example leads us to a very nice test for classifying critical points as relative maximums, relative minimums or neither.

In the graph above we can see that to the left of $x=-2$ the graph is decreasing and to the right of $x=-2$ the graph is increasing and $x=-2$ is a relative minimum. In other words, the graph is behaving around the minimum as we would expect it to. The same thing can be said for the relative maximum at $x=4$. The graph is increasing of the left and decreasing on the right. Finally, the graph is increasing on both sides of $x=0$ and so this critical point can't be a minimum or a maximum.

This can be summarized up in the following test.

First Derivative Test

Suppose that $x=c$ is a critical point of $f(x)$.

1. If $f'(x)>0$ to the left of $x=c$ and $f'(x)<0$ to the right of $x=c$ then $x=c$ is a relative maximum.
2. If $f'(x)<0$ to the left of $x=c$ and $f'(x)>0$ to the right of $x=c$ then $x=c$ is a relative

minimum.

3. If $f'(x)$ is the same sign on both sides of $x=c$ then $x=c$ is neither a relative maximum nor a relative minimum.

Example 3 Find and classify all the critical points of the following function. Give the intervals where the function is increasing and decreasing.

$$g(t) = t\sqrt[3]{t^2-4}$$

Solution

First we'll need the derivative so we can get our hands on the critical points.

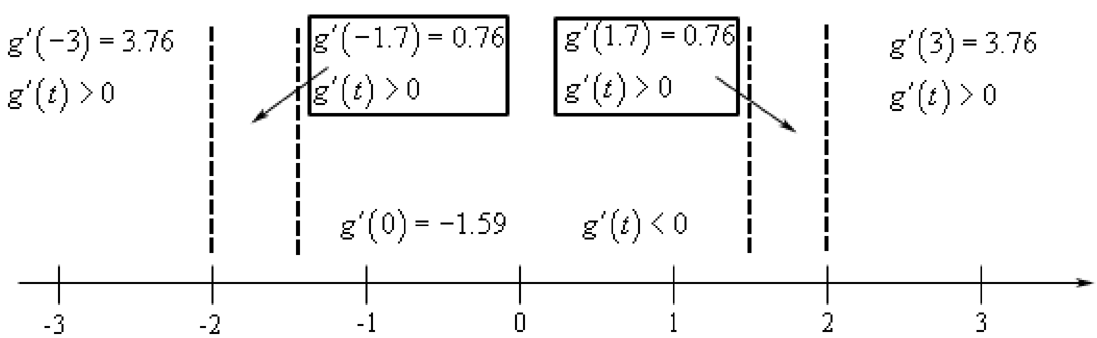
$$\begin{aligned} g'(t) &= \left(t^2-4\right)^{\frac{1}{3}} + \frac{2}{3}t^2\left(t^2-4\right)^{-\frac{2}{3}} \\ &= \left(t^2-4\right)^{\frac{1}{3}} + \frac{2t^2}{3\left(t^2-4\right)^{\frac{2}{3}}} \\ &= \frac{3\left(t^2-4\right)+2t^2}{3\left(t^2-4\right)^{\frac{2}{3}}} \\ &= \frac{5t^2-12}{3\left(t^2-4\right)^{\frac{2}{3}}} \end{aligned}$$

So, it looks like we'll have four critical points here. They are,

$t = \pm 2$ The derivative doesn't exist here.

$t = \pm\sqrt{\dfrac{12}{5}} = \pm 1.549$ The derivative is zero here.

Finding the intervals of increasing and decreasing will also give the classification of the critical points so let's get those first. Here is a number line with the critical points graphed and test points.



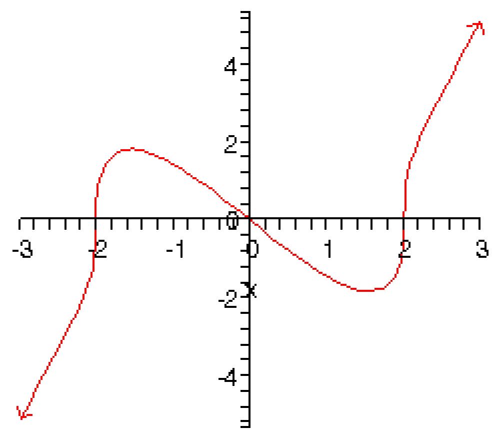
So, it looks like we've got the following intervals of increasing and decreasing.

$$\text{Increase}: \quad -\infty < x < -\sqrt{\frac{12}{5}} \text{ and } \sqrt{\frac{12}{5}} < x < \infty$$

$$\text{Decrease}: \ -\sqrt{\frac{12}{5}} < x < \sqrt{\frac{12}{5}}$$

From this it looks like *t=-2* and *t=2* are neither relative minimum or relative maximums since the function is increasing on both side of them. On the other hand, $t = -\sqrt{\frac{12}{5}}$ is a relative maximum and $t = \sqrt{\frac{12}{5}}$ is a relative minimum.

For completeness sake here is the graph of the function.



In the previous example the two critical points where the derivative didn't exist ended up not being relative extrema. Do not read anything into this. They often will be relative extrema. Check out this example in the Absolute Extrema to see an example of one such critical point.

Let's work a couple more examples.

Example 4 Suppose that the elevation above sea level of a road is given by the following function.

$$E(x) = 500 + \cos\left(\frac{x}{4}\right) + \sqrt{3}\sin\left(\frac{x}{4}\right)$$

where *x* is in miles. Assume that if *x* is positive we are to the east of the initial point of measurement and if *x* is negative we are to the west of the initial point of measurement.

If we start 25 miles to the west of the initial point of measurement and drive until we are 25 miles east of the initial point how many miles of our drive were we driving up an incline?

Solution

Okay, this is just a really fancy way of asking what the intervals of increasing and decreasing are for the function on the interval [-25,25]. So, we first need the derivative of the function.

$$E'(x) = -\frac{1}{4}\sin\left(\frac{x}{4}\right) + \frac{\sqrt{3}}{4}\cos\left(\frac{x}{4}\right)$$

Setting this equal to zero gives,

$$-\frac{1}{4}\sin\left(\frac{x}{4}\right) + \frac{\sqrt{3}}{4}\cos\left(\frac{x}{4}\right) = 0$$

$$\tan\left(\frac{x}{4}\right) = \sqrt{3}$$

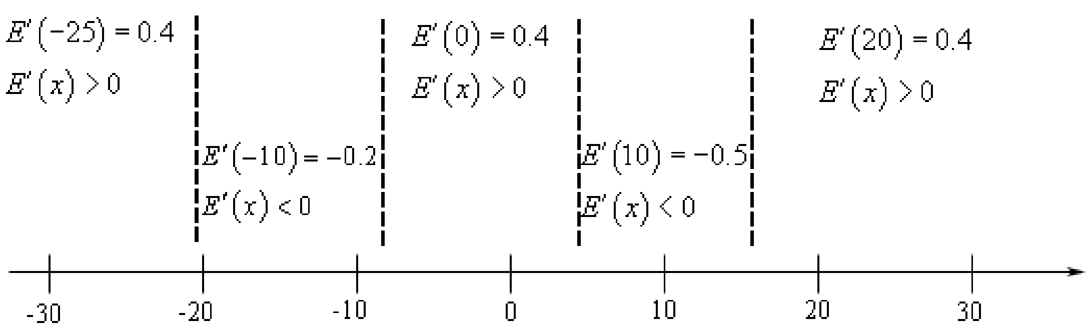
The solutions to this and hence the critical points are,

$$\begin{aligned} \frac{x}{4} &= \frac{\pi}{3} + 2\pi n, \ n = 0, \pm 1, \pm 2, \ldots \\ \frac{x}{4} &= \frac{4\pi}{3} + 2\pi n, \ n = 0, \pm 1, \pm 2, \ldots \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} x &= \frac{4\pi}{3} + 8\pi n, \ n = 0, \pm 1, \pm 2, \ldots \\ x &= \frac{16\pi}{3} + 8\pi n, \ n = 0, \pm 1, \pm 2, \ldots \end{aligned}$$

I'll leave it to you to check that the critical points that fall in the interval that we're after are,

$$-\frac{20\pi}{3}, -\frac{8\pi}{3}, \frac{4\pi}{3}, \frac{16\pi}{3} \qquad \Rightarrow \qquad -20.94, \ -8.38, \ 4.19, \ 16.76$$

Here is the number line with the critical points and test points.



So, it looks like the intervals of increasing and decreasing are,

$$\text{Increase : } \quad -25 < x < -\frac{20\pi}{3}, \ -\frac{8\pi}{3} < x < \frac{4\pi}{3} \text{ and } \frac{16\pi}{3} < x < 25$$

$$\text{Decrease : } \ -\frac{20\pi}{3} < x < -\frac{8\pi}{3} \text{ and } \frac{4\pi}{3} < x < \frac{16\pi}{3}$$

Notice that we had to end our intervals at -25 and 25 since we've done no work outside of these points and so we can't really say anything about the function outside of the interval [-25,25].

From the intervals of we can actually answer the question. We were driving on an incline during the intervals of increasing and so the total number of miles is,

$$\text{Distance } = \left(-\frac{20\pi}{3} - (-25)\right) + \left(\frac{4\pi}{3} - \left(-\frac{8\pi}{3}\right)\right) + \left(25 - \frac{16\pi}{3}\right) = 24.87 \text{ miles}$$

Even though the problem didn't ask for it we can also classify the critical points that are in the interval [-25,25].

$$\text{Relative Maximums} : -\frac{20\pi}{3}, \frac{4\pi}{3}$$

$$\text{Relative Minimums} : -\frac{8\pi}{3}, \frac{16\pi}{3}$$

Example 5 The population of rabbits (in hundreds) after *t* years in a certain area is given by the following function,

$$P(t) = t^2 \ln(3t) + 6$$

Determine if the population ever decreases in the first two years.

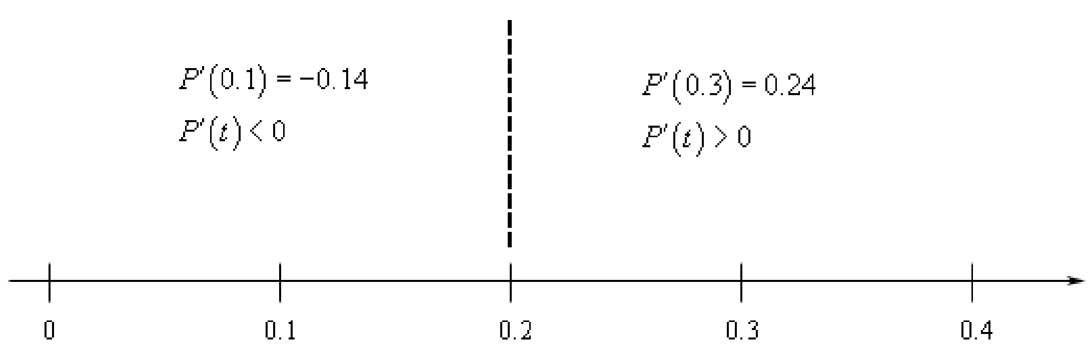
Solution

So, again we are really after the intervals and increasing and decreasing in the interval [0,2].

We found the only critical point to this function back in the Critical Points section to be,

$$x = \frac{1}{3\sqrt{\mathbf{e}}} = 0.202$$

Here is a number line for the intervals of increasing and decreasing.



So, it looks like the population will decrease for a short period and then continue to increase forever.

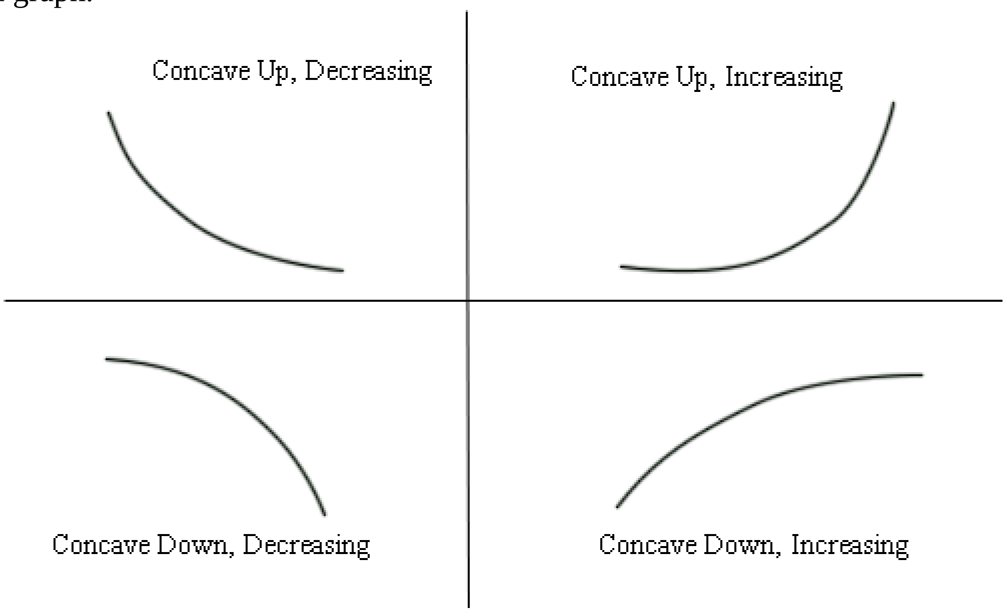
In this section we've seen how we can use the first derivative of a function to give us some information about the shape of a graph and how we can use this information in some applications.

Using the first derivative to give us information about a whether a function is increasing or decreasing is a very important application of derivatives and arises on a fairly regular basis in many areas.

The Shape of a Graph, Part II

In the previous section we saw how we could use the first derivative of a function to get some information about the graph of a function. In this section we are going to look at the information that the second derivative of a function can give us a about the graph of a function.

Before we do this we will need a couple of definitions out of the way. The main concept that we'll be discussing in this section is concavity. Concavity is usually best "defined" with a graph.



So a function is **concave up** if it opens up and the function is **concave down** if it opens down. Notice as well that concavity has nothing to do with increasing or decreasing. A function can be concave up and either increasing or decreasing. Similarly, a function can be concave down and either increasing or decreasing.

There's one more definition that we need to get out of the way.

Definition

A point $x=c$ is called an **inflection point** if the function is continuous at the point and the concavity of the graph changes at that point.

The following fact relates the second derivative of a function to the concavity of the function.

Fact

1. If $f''(x) > 0$ for all x in some interval then $f(x)$ is concave up on that interval.
2. If $f''(x) < 0$ for all x in some interval then $f(x)$ is concave down on that interval.

Notice that this means that a list of possible inflection points will be those points where the second is zero or doesn't exist.

With concavity as well as the increasing/decreasing information from the previous section we can get a pretty good idea of what a graph should look like.

Example 1 For the following function identify the intervals where the function is increasing and decreasing and the intervals where the function is concave up and concave down. Use this information to sketch the graph.

$$h(x) = 3x^5 - 5x^3 + 3$$

Solution

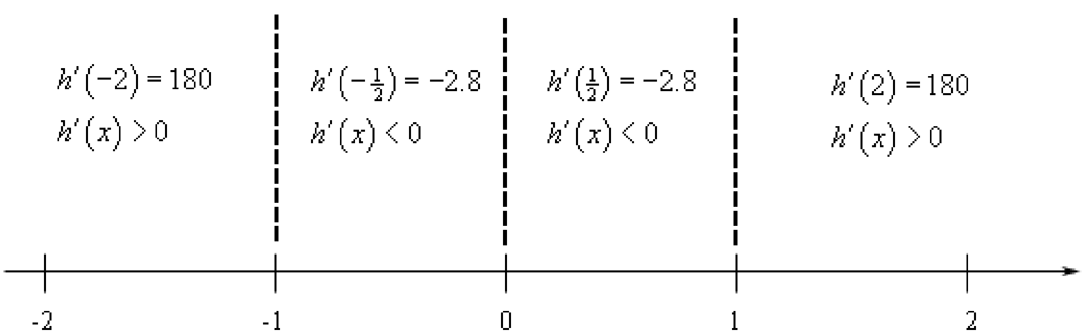
Okay, we are going to need the first two derivatives so let's get those first.

$$h'(x) = 15x^4 - 15x^2 = 15x^2(x-1)(x+1)$$

$$h''(x) = 60x^3 - 30x = 30x(2x^2 - 1)$$

Let's start with the increasing/decreasing information since we should be fairly comfortable with that after the last section.

There are three critical points for this function : $x=-1$, $x=0$, and $x=1$. Below is the number line for the increasing/decreasing information.



So, it looks like we've got the following intervals of increasing and decreasing.

$$\text{Increasing : } -\infty < x < -1 \text{ and } 1 < x < \infty$$

$$\text{Decreasing : } -1 < x < 1$$

Note that from the first derivative test we can also say that $x=-1$ is a relative maximum and that $x=1$ is a relative minimum. Also $x=0$ is neither a relative minimum or maximum.

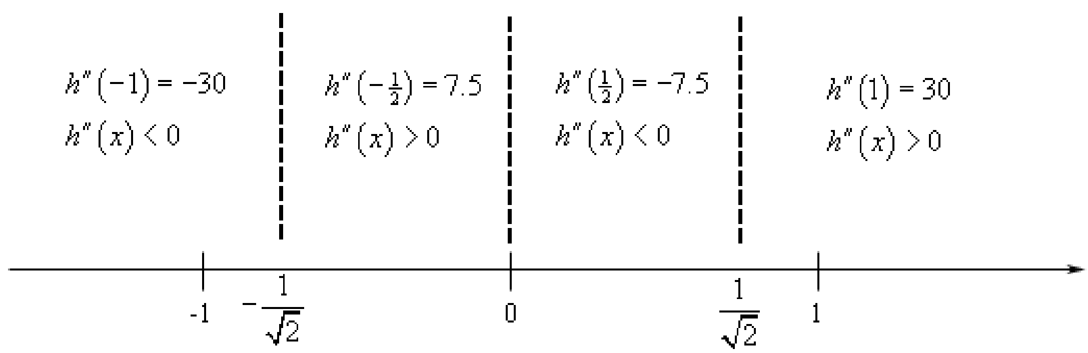
Now let’s get the intervals where the function is concave up and concave down. If you think about it this process is almost identical to the process we use to identify the intervals of increasing and decreasing. This only difference is that we will be using the second derivative instead of the first derivative.

The first thing that we need to do is identify the possible inflection points. These will be where the second derivative is zero or doesn’t exist. The second derivative in this case is a polynomial and so will exist everywhere. It will be zero at the following points.

$$x = 0, \; x = \pm\frac{1}{\sqrt{2}} = \pm 0.7071$$

As with the increasing and decreasing part we can draw a number line up and use these points to divide the number line into regions. In these regions we know that the second derivative will always have the same value since these three points are the only places where the function *may* change sign. Therefore, all that we need to do is pick a point from each region and plug it into the second derivative. The second derivative will then have that sign in the whole region from which the point came from

Here is the number line for this second derivative.



So, it looks like we’ve got the following intervals of concavity.

$$\text{Concave Up : } -\frac{1}{\sqrt{2}} < x < 0 \text{ and } \frac{1}{\sqrt{2}} < x < \infty$$

$$\text{Concave Down : } -\infty < x < -\frac{1}{\sqrt{2}} \text{ and } 0 < x < \frac{1}{\sqrt{2}}$$

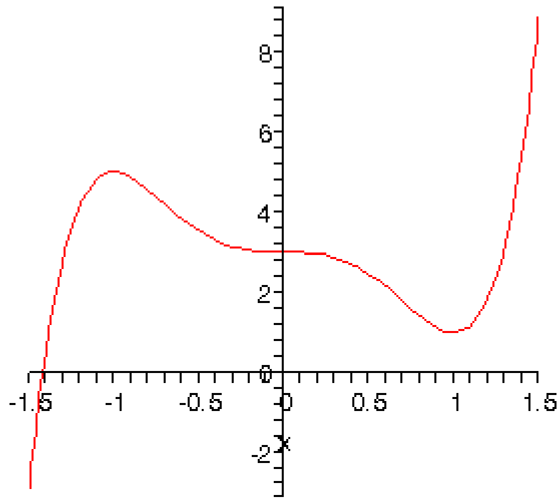
This also means that

$$x = 0, \; x = \pm\frac{1}{\sqrt{2}} = \pm 0.7071$$

are all inflection points.

All this information can be a little overwhelming when going to sketch the graph. The first thing that we should do is get some starting points. The critical points and inflection points are good starting points. So, first graph these points. Now, start to the left and start graphing the increasing/decreasing information. As we graph this we will make sure that the concavity information matches up with what we're graphing.

Using all this information to sketch the graph gives the following graph.



We can use the previous example to get another way to classify some of the critical points as relative maximums or relative minimums.

Notice that $x=-1$ is a relative maximum and that the function is concave down at this point. This means that $f''(-1)$ must be negative. Likewise, $x=1$ is a relative minimum and the function is concave up at this point. This means that $f''(1)$ must be positive.

As we'll see we will need to be very careful with $x=0$. In this case the second derivative is zero, but that will not actually mean that $x=0$ is not a relative minimum or maximum.

Also, all of the critical points in this example were critical points in which the first derivative were zero and this is required. We will not be able to use this test on critical points where the derivative doesn't exist.

Here is the test.

Second Derivative Test

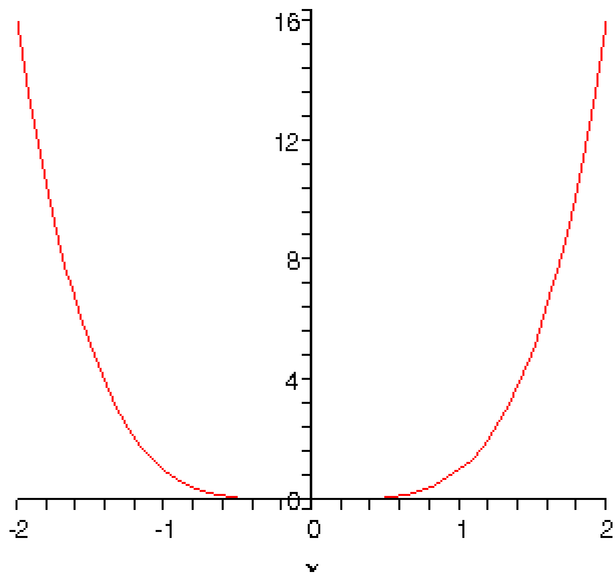
Suppose that $x=c$ is a critical point of $f(x)$ such that $f'(c)=0$. Then,

1. If $f''(c)<0$ $x=c$ is a relative maximum.
2. If $f''(c)>0$ $x=c$ is a relative minimum.

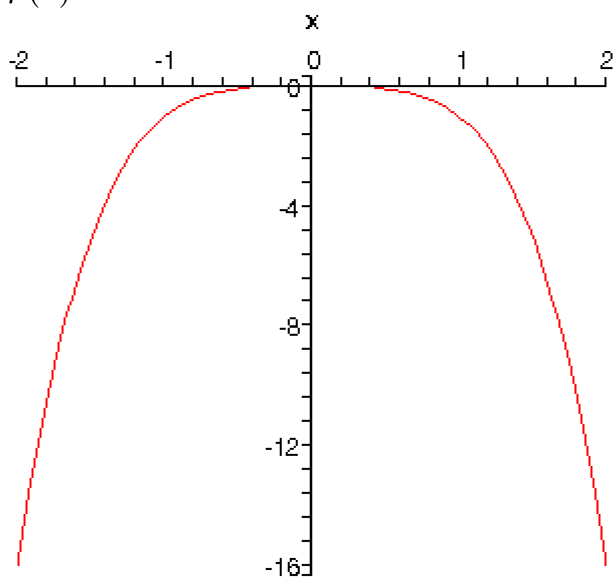
3. If $f''(c)=0$ $x=c$ can be a relative maximum, relative minimum or neither.

The third part of the second derivative test is important to notice. If the second derivative is zero then the critical point can be anything. Below are the graphs of three functions all of which have a critical point at $x=0$, the second derivative of all of the functions is zero and yet all three possibilities are exhibited.

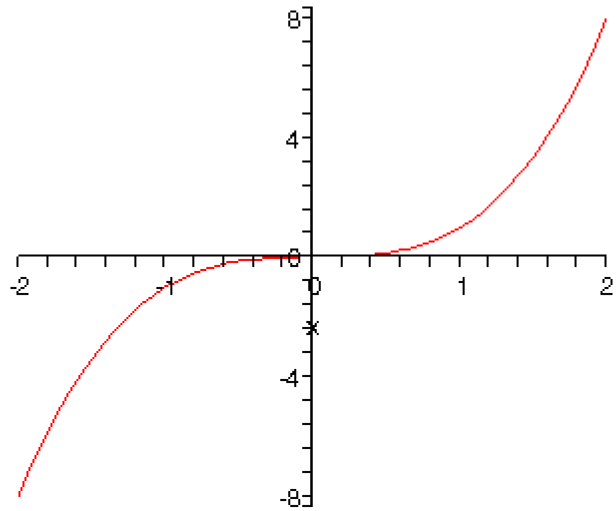
The first is the graph of $f(x)=x^4$. This graph has a relative minimum at $x=0$.



Next is the graph of $f(x)=-x^4$ which has a relative maximum at $x=0$.



Finally, there is the graph of $f(x) = x^3$ and this graph had neither a relative minimum or a relative maximum at *x=0*.



So, we can see that we have to be careful if we fall into the third case. For those times when we do fall into this case we will have to resort to other methods of classifying the critical point. This is usually done with the first derivative test.

Example 2 Use the second derivative test to classify the critical points of the function,

$$h(x) = 3x^5 - 5x^3 + 3$$

Solution

Note that all we're doing here is verifying the results from the first example. The second derivative is,

$$h''(x) = 60x^3 - 30x$$

The value of the second derivative at the three critical points (*x=-1*, *x=0*, and *x=1*) are,

$$h''(-1) = -30 \qquad h''(0) = 0 \qquad h''(1) = 30$$

The second derivative at *x=-1* is negative so this critical point this is a relative maximum as we saw in the first example. The second derivative at *x=1* is positive and so we have a relative minimum here as we also saw in the first example.

In the case of *x=0* the second derivative is zero and *in this case* the critical point is not a relative extrema.

Let's work one more example.

Example 3 For the following function find the inflection points and use the second derivative test, if possible, to classify the critical points.

$$f(t) = t(6-t)^{\frac{2}{3}}$$

Solution

We'll need the first and second derivatives to get us started.

$$f'(t) = \frac{18-5t}{3(6-t)^{\frac{1}{3}}} \qquad\qquad f''(t) = \frac{10t-72}{9(6-t)^{\frac{4}{3}}}$$

The critical points are,

$$t = \frac{18}{5} = 3.6 \qquad\qquad t = 6$$

Notice as well that we won't be able to use the second derivative test on *t=6* to classify this derivative since this is a critical point in which the derivative doesn't exist. If you look at the second derivative you can see why we need to exclude this critical point. The second derivative doesn't exist at *t=6* so we can't plug this into the second derivative as is required by the test.

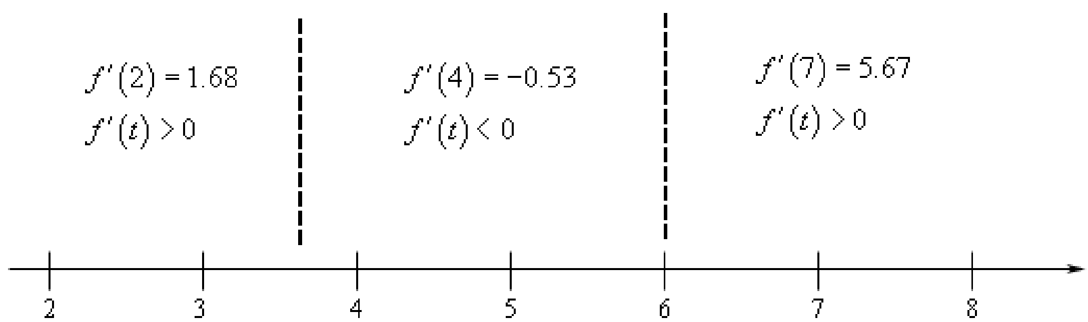
So, let's classify the other critical point.

$$f''\left(\frac{18}{5}\right) = -1.245 < 0$$

According to the second derivative test this should be a relative maximum.

Now, even though this isn't asked for in the problem let's classify *t=6* using the first derivative test. This will also verify the second derivative test for the first critical point.

Here is the number line for the first derivative.

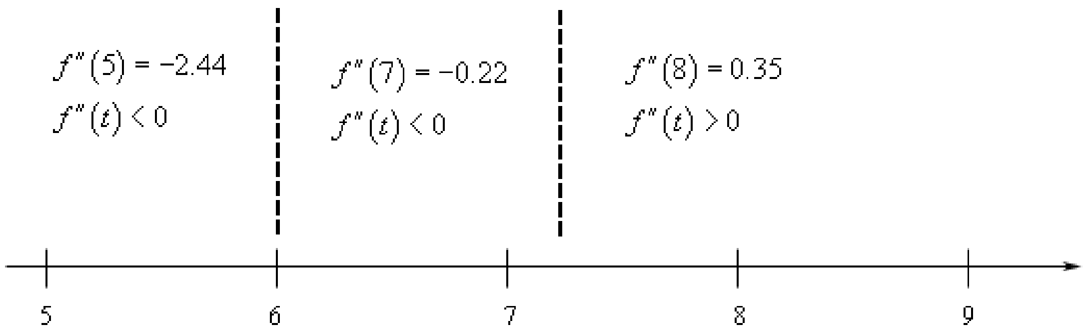


So, according to the first derivative test we got the first critical point correct it is a relative maximum. The second critical point (*t=6*) is a relative minimum. So, be careful not to assume that a critical point that can't be used in the second derivative test won't be a relative extrema.

Okay, let's finish the problem out. We will need the list of possible inflection points. These are,

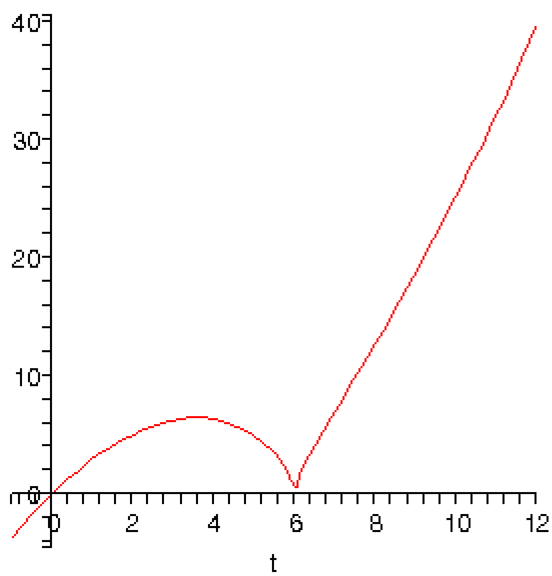
$$t = 6 \qquad\qquad t = \frac{72}{10} = 7.2$$

Here is the number line for the second derivative. Note that we will need this to see if the two points above are in fact inflection points.



So, the concavity only changes at *t=7.2* and so this is the only inflection point for this function.

For the sake of completeness here is the graph of the function.



The inflection point at *t=7.2* is hard to see, but it is there.

The Mean Value Theorem

There isn't a whole lot to this section. We're just going to talk a little bit about the Mean Value Theorem.

Mean Value Theorem

Suppose *f(x)* is a function that satisfies both of the following.

1. $f(x)$ is continuous on the closed interval $[a,b]$.
2. $f(x)$ is differentiable on the open interval (a,b).

Then there is a number c such that $a < c < b$ and

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Or,

$$f(b) - f(a) = f'(c)(b - a)$$

Note that the Mean Value Theorem doesn't tell us what c is. It only tells us that there is at least one number c that will satisfy the conclusion of the theorem.

Example 1 Determine all the numbers c which satisfy the conclusions of the Mean Value Theorem for the following function.

$$f(x) = x^3 + 2x^2 - x \quad \text{on} \quad [-1, 2]$$

Solution

There isn't really a whole lot to this problem other than to notice that since $f(x)$ is a polynomial it is both continuous and differentiable (*i.e.* the derivative exists) on the interval given.

First let's find the derivative.

$$f'(x) = 3x^2 + 4x - 1$$

Now, to find the numbers that satisfy the conclusions of the Mean Value Theorem all we need to do is plug this into the formula given by the Mean Value Theorem.

$$f'(c) = \frac{f(2) - f(-1)}{2 - (-1)}$$

$$3c^2 + 4c - 1 = \frac{14 - 2}{3} = \frac{12}{3} = 4$$

Now, this is just a quadratic equation,

$$3c^2 + 4c - 1 = 4$$

$$3c^2 + 4c - 5 = 0$$

Using the quadratic formula on this we get,

$$c = \frac{-4 \pm \sqrt{16 - 4(3)(-5)}}{6} = \frac{-4 \pm \sqrt{76}}{6}$$

So, solving gives two values of c.

$$c = \frac{-4 + \sqrt{76}}{6} = 0.7863 \qquad c = \frac{-4 - \sqrt{76}}{6} = -2.1196$$

Notice that only one of these is actually in the interval given in the problem. That means that we will exclude the second one (since it isn't in the interval). The number that we're after in this problem is,

$$c = 0.7863$$

Be careful to not assume that only one of the numbers will work. It is possible for both of them to work.

Let's take a look at an application of the Mean Value Theorem.

Example 2 Suppose that we know that *f(x)* is continuous and differentiable. Let's also suppose that we know that $f(6) = -2$ and that we know that $f'(x) \le 10$. What is the largest possible value for $f(15)$?

Solution
Let's start with the conclusion of the Mean Value Theorem.

$$f(15) - f(6) = f'(c)(15-6)$$

Plugging in for the known quantities and rewriting this a little gives,

$$f(15) = f(6) + f'(c)(15-6) = -2 + 9f'(c)$$

Now we know that $f'(x) \le 10$ so in particular we know that $f'(c) \le 10$. This gives us the following,

$$\begin{aligned} f(15) &= -2 + 9f'(c) \\ &\le -2 + (9)10 \\ &= 88 \end{aligned}$$

All we did was replace $f'(c)$ with its largest possible value.

This means that the largest possible value for $f(15)$ is 88.

Here are a couple of nice facts that can be proved using the Mean Value Theorem. Note that in both of these facts we are assuming the functions are continuous and differentiable on the interval (a,b).

Fact 1

If $f'(x) = 0$ for all *x* in an interval (a,b) then $f(x)$ is constant on (a,b).

This fact is very easy to prove so let's do that. Take any two *x*'s in the interval (a,b), say x_1 and x_2. Then since $f(x)$ is continuous and differential on (a,b) it must also be continuous and differentiable on (x_1,x_2). This means that we can apply the Mean Value Theorem for these two values of *x*. Doing this gives,

$$f(x_2)-f(x_1)=f'(c)(x_2-x_1)$$

where $x_1<c<x_2$. But by assumption $f'(x)=0$ for all *x* in an interval (a,b) and so in particular we must have,

$$f'(c)=0$$

Putting this into the equation above gives,

$$f(x_2)-f(x_1)=0 \qquad \Rightarrow \qquad f(x_2)=f(x_1)$$

Now, since x_1 and x_2 where any two values of *x* in the interval (a,b) we can see that we must have $f(x_2)=f(x_1)$ for all x_1 and x_2 in the interval and this is exactly what it means for a function to be constant on the interval and so we've proven the fact.

Fact 2

If $f'(x)=g'(x)$ for all *x* in an interval (a,b) then in this interval we have $f(x)=g(x)+c$ where *c* is some constant.

This fact is a direct result of the previous fact and is also easy to prove.

If we first define,

$$h(x)=f(x)-g(x)$$

Then since both $f(x)$ and $g(x)$ are continuous and differentiable in the interval (a,b) then so must be $h(x)$. Therefore the derivative of $h(x)$ is,

$$h'(x)=f'(x)-g'(x)$$

However, by assumption $f'(x)=g'(x)$ for all *x* in an interval (a,b) and so we must have that $h'(x)=0$ for all *x* in an interval (a,b). So, by Fact 1 $h(x)$ must be constant on the interval.

This means that we have,

$$\begin{aligned} h(x)&=c \\ f(x)-g(x)&=c \\ f(x)&=g(x)+c \end{aligned}$$

which is what we where trying to show.

Optimization

In this section we are going to look at optimization problems. In optimization problems, in general, we are looking for the largest and/or smallest that a function can be. We saw how to one kind of optimization problem in the Absolute Extrema section where we found the largest and smallest value that a function would take on an interval.

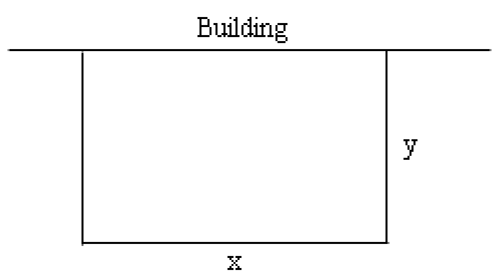
In this section we are going to look at another type of optimization problem. Here we will be looking for the largest of smallest values of a function subject to some kind of constraint. It's usually easiest to see how these work with some examples.

Example 1 We need to enclose a field with a fence. We have 500 feet of fencing material and a building is on one side of the field and so won't need any fencing. Determine the dimensions of the field that will enclose the largest area.

Solution

In all of these problems we will have two functions. The first is the function that we are actually trying to optimize and the second will be some constraint. The constraint will be an equation that must be true no matter what else is happening in the problem.

Let's sketch an image of what's going on. You should always do that with these problems.



In this problem we want to maximize the area of a field and we know that will use 500 ft of fencing material. So, the area will be the function we are trying to optimize and the amount of fencing is the constraint.

$$\begin{aligned} &\text{Maximize : } A = xy \\ &\text{Contraint : } 500 = x + 2y \end{aligned}$$

Okay, we know how to find the largest or smallest value of a function provided it's only got a single variable. The area function (as well as the constraint) has two variables in it and so what we know about finding absolute extrema won't work. However, if we solve the constraint for one of the two variables we can substitute this into the area and we will then have a function of a single variable.

So, let's solve the constraint for x. Note that we could have just as easily solved for y.

$$x = 500 - 2y$$

Substituting this into the area function gives a function of y.

$$A(y) = (500 - 2y)\, y = 500y - 2y^2$$

Now we want to find the largest value this will have on the interval [0,250]. Note that the interval corresponds to taking $y=0$ and $y=250$ (there's two sides and so if each is 250 ft we use the whole 500ft…) and these won't make any sense from a physical standpoint if we actually want to enclose some area. In both of these cases we would get an area of zero and so will clearly not maximize the area.

So, recall from the section on Absolute Extrema that the maximum will occur at critical points and/or end points. As we've already pointed out the end points in this case don't make any sense. That means our only option will be the critical points.

So let's get the derivative and find the critical points.

$$A'(y) = 500 - 4y$$

Setting this equal to zero and solving gives a lone critical point of $y=125$. Now, plugging this into the area gives an area of 31250 ft^2. So according to the method from Absolute Extrema section this must be the largest possible area, since the area at the endpoints is zero.

So, let's not forget to get the value of x and then we'll have the dimensions. We can get the x by plugging in our y into the constraint.

$$x = 500 - 2(125) = 250$$

The dimensions of the field that will give the largest area, subject to the fact that we used exactly 500 ft of fencing material, are 250 x 125.

Note that there is a nice quick check to see if we are one the right track for these problems as well. In the previous example, if we had also found the second derivative we would have,

$$A''(y) = -4$$

$$A''(125) = -4 < 0$$

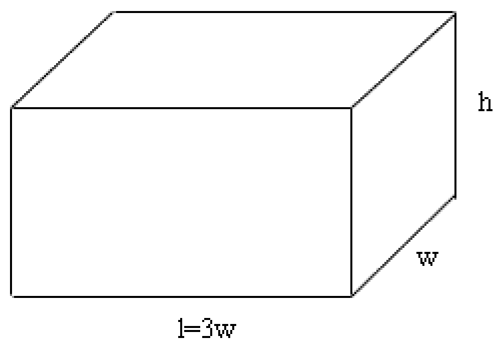
Then from the second derivative test we can see that our critical point is also a relative maximum. This is sometimes a nice check. If we had used the second derivative test here and found it to be a relative minimum we would have know that we had the wrong answer since there is no way that a relative minimum could ever maximize a quantity.

In almost all the remaining problems we will make use of this to fact to verify our results. Also, if we get multiple critical points this will be a good way to quickly eliminate all the critical points that can't possibly be answers.

Let's work another example.

Example 2 We are going to construct a box whose base length is 3 times the base width. The material used to build the top and bottom cost \$10/ft^2 and the material used to build the sides cost \$6/ft^2. If the box must have a volume of 50ft^3 determine the dimensions that will minimize the cost to build the box.

Solution
First, a quick figure.



Okay, we want to minimize the cost subject to the constraint that the volume must be 50ft^3. Note as well that the cost is just the area of each side times the appropriate cost.

The two functions we'll be working with here this time are,

$$\begin{aligned} \text{Minimize : } C &= 10(2lw) + 6(2wh + 2lh) \\ &= 60w^2 + 48wh \\ \text{Constraint : } 50 &= lwh = 3w^2 h \end{aligned}$$

As with the first example, we will solve the constraint for one of the variables and plug this into the cost. It will definitely be easier to solve the constraint for *h* so let's do that.

$$h = \frac{50}{3w^2}$$

Plugging this into the cost gives,

$$C(w) = 60w^2 + 48w\left(\frac{50}{3w^2}\right)$$
$$= 60w^2 + \frac{800}{w}$$

Now, let's get the first and second derivatives,

$$C'(w) = 120w - 800w^{-2} = \frac{120w^3 - 800}{w^2}$$
$$C''(w) = 120 + 1600w^{-3}$$

So, it look's like we've got two critical points here. The first is obvious, $w=0$, and it's also just as obvious that this will not be the correct one. We are building a box here and w is the box's width and so since it makes no sense to talk about a box with zero width we will ignore this critical point. The next critical point will come from determining where the numerator is zero.

$$120w^3 - 800 = 0 \qquad \Rightarrow \qquad w = \sqrt[3]{\frac{800}{120}} = \sqrt[3]{\frac{20}{3}} = 1.882$$

So, once we throw out $w=0$, we've got a single critical point. Let's run it through the second derivative test real quickly to make sure that the critical point is a relative minimum.

$$C''(1.882) = 120 + \frac{1600}{(1.882)^3} > 0$$

We don't so much care what the value is. All that we want to know is that it's positive and so the critical point will be a relative minimum.

Now, since this is the only critical point and it's a relative minimum it must also be the smallest value the cost function will take.

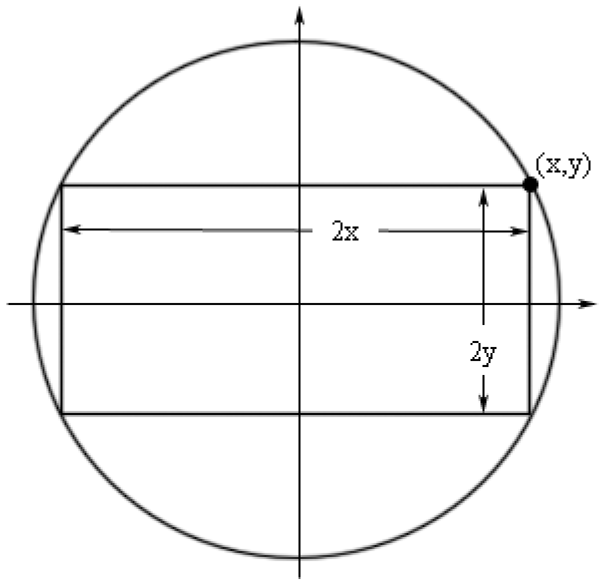
All we need to do now is to find the remaining dimensions.

$$w = 1.882$$
$$l = 3w = 3(1.882) = 5.646$$
$$h = \frac{50}{3w^2} = \frac{50}{3(1.882)^2} = 4.706$$

Example 3 Determine the area of the largest rectangle that can be inscribed in a circle of radius 4.

Solution

Huh? This problem is best described with a sketch. Here is what we're looking for.



We want the area of the largest rectangle that we can fit inside a circle and have all of its corners touching the circle.

To do this problem it's easiest to assume that everything is centered at the origin. Doing this we know that the equation of the circle will be

$$x^2 + y^2 = 16$$

and that the right upper corner of the rectangle will have the coordinates *(x,y)*. This means that the width of the rectangle will be $2x$ and the height of the rectangle will be $2y$. The area of the rectangle will then be,

$$A = (2x)(2y) = 4xy$$

So, we've got the function we want to maximize (the area), but what is the constraint? Well since the coordinates of the upper right corner must be on the circle we know that x and y must satisfy the equation of the circle. In other words, the equation of the circle is the constraint.

The first thing to do then is to solve the constraint for one of the variables.

$$y = \pm\sqrt{16 - x^2}$$

Since the point that we're looking at is in the first quadrant we know that y must be positive and so we can take the "+" part of this. Plugging this into the area and computing the first derivative gives,

$$A(x) = 4x\sqrt{16 - x^2}$$

$$A'(x) = 4\sqrt{16 - x^2} - \frac{4x^2}{\sqrt{16 - x^2}} = \frac{64 - 8x^2}{\sqrt{16 - x^2}}$$

Computing the second derivative in this case would overly complicate the problem so we won't bother with that this time. Also notice that the largest that x could be is $x=4$ and the smallest is $x=0$ and in both of these cases the area will be zero and so won't give the largest rectangle. All we need to do now is determine the critical points that fall between $x=0$ and $x=4$.

We get four critical points, two from the numerator and two from the denominator.

$$16 - x^2 = 0 \qquad \Rightarrow \qquad x = \pm 4$$

$$64 - 8x^2 = 0 \qquad \Rightarrow \qquad x = \pm 2\sqrt{2}$$

We are assuming that the point is in the first quadrant so we can ignore the negative values. Also, we've already excluded $x=4$ since that is one of the end points. This leaves one critical point.

$$x = 2\sqrt{2}$$

The area for the rectangle with this value of x is then $A=32$. Now, from the Absolute Extrema section we know that the largest and smallest value of the area will occur at a critical point or the end points. Since this is clearly larger than the area at the endpoints we've got the largest area.

All the problem asked for was the area of the rectangle, but lets' go ahead and finish this problem out with find the second dimension of the rectangle as well. To do this all we need to do is plug the critical point into the equation for y we found above to see that,

$$y = 2\sqrt{2}$$

So, it looks like the largest area actually comes for a square in this case.

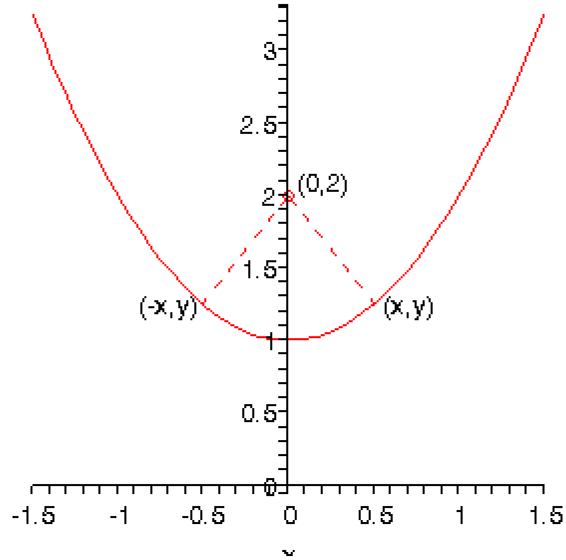
In the last two examples we have excluded critical points from the problem. Do not get into the habit of this. In both of these cases we had a physical reason for doing this. Often physical reasons will allow us to exclude critical points, however, we won't always be able to do this so don't get in the habit of automatically excluding certain critical points (in particular zero and negative critical points). We should always get all possible critical points and then decide if there is some reason (physical or otherwise) for excluding any of them.

Let's work one more example.

Example 4 Determine the point(s) on $y = x^2 + 1$ that are closest to (0,2).

Solution

Again, let's get a quick sketch.



So, we're looking for the shortest length of the dashed line. Notice as well that if the shortest distance isn't at $x=0$ there will be two points on this graph as we've shown that will give the shortest distance. This is coming about because the parabola is symmetric to the y-axis and the point in question is on the y-axis. This won't always be the case of course so don't always expect two points.

In this case we need to minimize the distance between the point (0,2) and any point that is one the graph (x,y). Or,

$$d = \sqrt{(x-0)^2 + (y-2)^2} = \sqrt{x^2 + (y-2)^2}$$

If you think about the situation here it makes sense that the point that minimizes the distance will also minimize the square of the distance and so since it will be easier to work with we will use the square of the distance and minimize that. So, the function that we're going to minimize is,

$$D = d^2 = x^2 + (y-2)^2$$

The constraint in this case is the function itself since the point must lie on the graph of the function.

At this point there are two methods for proceeding. One of which will require significantly more work than the other. Let's take a look at both of them.

Solution 1
In this case we will use the constraint in probably the most obvious way. We already have the constraint solved for y so let's plug that into the square of the distance and get the derivatives.

$$D(x) = x^2 + \left(x^2 + 1 - 2\right)^2 = x^4 - x^2 + 1$$
$$D'(x) = 4x^3 - 2x = 2x\left(2x^2 - 1\right)$$
$$D''(x) = 12x^2 - 2$$

So, it looks like there are three critical points for the square of the distance.

$$x = 0, \quad x = \pm\frac{1}{\sqrt{2}}$$

Before going any farther, let's check these in the second derivative to see if they are all relative minimums.

$$D''(0) = -2 < 0 \qquad D''\left(\frac{1}{\sqrt{2}}\right) = 4 \qquad D''\left(-\frac{1}{\sqrt{2}}\right) = 4$$

So, *x=0* is relative maximum and so can't possibly be the minimum distance. That means that we've got two critical points. Note that this makes sense given the initial discussion on how many we could expect to get. However, let's be careful and plug both of these into the square of the distance and see what we get.

$$D\left(\frac{1}{\sqrt{2}}\right) = \frac{3}{4} \qquad D\left(-\frac{1}{\sqrt{2}}\right) = \frac{3}{4}$$

Since both give the same value of the square of the distance and they are relative minimums they are both valid answers.

All that we need to do is to find the value of *y* for these points.

$$x = \frac{1}{\sqrt{2}} \quad : \quad y = \frac{3}{2}$$
$$x = -\frac{1}{\sqrt{2}} : \quad y = \frac{3}{2}$$

So, the points on the graph that are closest to (0,2) are,

$$\left(\frac{1}{\sqrt{2}}, \frac{3}{2}\right) \qquad \left(-\frac{1}{\sqrt{2}}, \frac{3}{2}\right)$$

Solution 2
The first solution that we worked was actually the long solution. There is a much shorter solution to this problem. Instead of plugging *y* into the square of the distance let's plug in *x*. From the constraint we get,

$$x^2 = y - 1$$

and notice that the only place *x* show up in the square of the distance it shows up as x^2 and let's just plug this into the square of the distance. Doing this gives,

$$D(y) = y - 1 + (y-2)^2 = y^2 - 3y + 3$$
$$D'(y) = 2y - 3$$
$$D''(y) = 2$$

There is now a single critical point, $y = \frac{3}{2}$, and from the second derivative it is guaranteed to be a relative minimum. So all that we need to do at this point is find the value(s) of *x* that go with this value of *y*.

$$x^2 = \frac{3}{2} - 1 = \frac{1}{2} \qquad \Rightarrow \qquad x = \pm\frac{1}{\sqrt{2}}$$

The points are then,

$$\left(\frac{1}{\sqrt{2}}, \frac{3}{2}\right) \qquad \left(-\frac{1}{\sqrt{2}}, \frac{3}{2}\right)$$

So, for significantly less work we got exactly the same answer.

In this last example we saw that sometimes there is more than one solution method and that sometimes the simplest is not always the most obvious. We should always try to think ahead with these problems and see if there might be a simpler method to work the problem.

Indeterminate Forms and L'Hospital's Rule

Back in the chapter on Limits we saw methods for dealing with the following limits.

$$\lim_{x \to 4} \frac{x^2 - 16}{x - 4} \qquad \lim_{x \to \infty} \frac{4x^2 - 5x}{1 - 3x^2}$$

In the first limit if we plugged in *x=4* we would get 0/0 and in the second limit if we "plugged" in infinity we would get $\infty / -\infty$ (recall that as *x* goes to infinity a polynomial will behave in the same fashion that it's largest power behaves). Both of these are called **indeterminate forms**. In both of these cases there are competing interests or rules and it's not clear which will win out.

In the case of 0/0 we typically think of a fraction that has a numerator of zero as being zero. However, we also tend to think of fractions in which the denominator is going to zero as infinity or might not exist at all. Likewise, we tend to think of a fraction in which the numerator and denominator are the same as one. So, which will win out? Or will neither win out and they all "cancel out" and the limit will reach some other value?

In the case of $\infty / -\infty$ we have a similar set of problems. If the numerator of a fraction is going to infinity we tend to think of the whole fraction going to infinity. Also if the denominator is going to infinity we tend to think of the fraction as going to zero. We also have the case of a fraction in which the number and denominator are the same (ignoring the minus sign) and so we might get -1. Again, it's not clear which of these will win out, if any of them will win out.

With the second limit there is the further problem that infinity isn't really a number and so we really shouldn't even treat it like a number. Much of the time it simply won't behave as we would expect it to if it was a number. To look a little more into this check out the Types of Infinity section in the Extras chapter at the end of this document.

This is the problem with indeterminate forms. It's just not clear what is happening in the limit. There are other types of indeterminate forms as well. Some other types are,

$$(0)(\pm\infty) \qquad 1^{\infty} \qquad 0^{0} \qquad \infty^{0} \qquad \infty-\infty$$

These all have competing interests or rules that tell us what should happen and it's just not clear which, if any, of the interests or rules will win out. The topic of this section is how to deal with these kinds of limits.

As already pointed out we do know how to deal with some kinds of indeterminate forms already. For the two limits above we work them as follows.

$$\lim_{x\to 4}\frac{x^2-16}{x-4}=\lim_{x\to 4}(x+4)=8$$

$$\lim_{x\to\infty}\frac{4x^2-5x}{1-3x^2}=\lim_{x\to\infty}\frac{4-\frac{5}{x}}{\frac{1}{x^2}-3}=-\frac{4}{3}$$

In the first case we simply factored, canceled and took the limit and in the second case we factored out an x^2 from both the numerator and the denominator and took the limit. Notice as well that none of the competing interests or rules in these cases won out! That is often the case.

So we can deal with some of these. However what about the following two limits.

$$\lim_{x\to 0}\frac{\sin x}{x} \qquad\qquad \lim_{x\to\infty}\frac{\mathbf{e}^x}{x^2}$$

This first is a 0/0 indeterminate form, but we can't factor this one. The second is a ∞/∞ indeterminate form, but we can't just factor an x^2 out of the numerator. So, nothing that we've got in our bag of tricks will work with these two limits.

This is where the subject of this section comes into play.

L'Hospital's Rule

Suppose that we have one of the following cases,

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0} \qquad \text{OR} \qquad \lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\pm\infty}{\pm\infty}$$

where *a* can be infinity or negative infinity. In these cases we have,

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

So, L'Hospital's Rule tells us that if we have an indeterminate form 0/0 or ∞/∞ all we need to do is differentiate the numerator and differentiate the denominator and then take the limit.

Let's work some examples.

Example 1 Evaluate each of the following limits.

(a) $\lim\limits_{x\to 0}\frac{\sin x}{x}$

(b) $\lim\limits_{t\to 1}\frac{5t^4-4t^2-1}{10-t-9t^3}$

(c) $\lim\limits_{x\to\infty}\frac{\mathbf{e}^x}{x^2}$

Solution

(a) So, we have already established that this is a 0/0 indeterminate form so let's just apply L'Hospital's Rule.

$$\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{\cos x}{1}=\frac{1}{1}=1$$

(b) In this case we also have a 0/0 indeterminate form and if we were really good at factoring we could factor the numerator and denominator, simplify and take the limit. However, that's going to be more work than just using L'Hospital's Rule.

$$\lim_{t\to 1}\frac{5t^4-4t^2-1}{10-t-9t^3}=\lim_{t\to 1}\frac{20t^3-8t}{-1-27t^2}=\frac{20-8}{-1-27}=-\frac{3}{7}$$

(c) This was the other limit that we started off looking at and we know that it's the indeterminate form ∞/∞ so let's apply L'Hospital's Rule.

$$\lim_{x\to\infty}\frac{\mathbf{e}^x}{x^2}=\lim_{x\to\infty}\frac{\mathbf{e}^x}{2x}$$

Now we have a small problem. This new limit is also a ∞/∞ indeterminate form. However, it's not really a problem. We know how to deal with these kinds of limits. Just apply L'Hospital's Rule.

$$\lim_{x\to\infty}\frac{\mathbf{e}^x}{x^2}=\lim_{x\to\infty}\frac{\mathbf{e}^x}{2x}=\lim_{x\to\infty}\frac{\mathbf{e}^x}{2}=\infty$$

Sometimes we will need to apply L'Hospital's Rule more than once.

L'Hospital's Rule is works great on the two indeterminate forms 0/0 and $\pm\infty/\pm\infty$. However, there are many more indeterminate forms out there as we saw earlier. Let's take a look at some of those and see how we deal with those kinds of indeterminate forms.

We'll start with the indeterminate form $(0)(\pm\infty)$.

Example 2 Evaluate the following limit.

$$\lim_{x\to 0^+} x\ln x$$

Solution

So, in the limit, we get the indeterminate form $(0)(-\infty)$. L'Hospital's Rule won't work on products, it only works on quotients. However, we can turn this into a faction if we write things a little.

$$\lim_{x\to 0^+} x\ln x=\lim_{x\to 0^+}\frac{\ln x}{1/x}$$

The function is the same, just rewritten, and the limit is now in the form $-\infty/\infty$ and we can now use L'Hospital's Rule.

$$\lim_{x\to 0^+} x\ln x=\lim_{x\to 0^+}\frac{\ln x}{1/x}=\lim_{x\to 0^+}\frac{1/x}{-1/x^2}$$

Now, this is a mess, but it cleans up nicely.

$$\lim_{x\to 0^+} x\ln x=\lim_{x\to 0^+}\frac{1/x}{-1/x^2}=\lim_{x\to 0^+}-x=0$$

In the previous example we used the fact that we can always write a product of functions as a quotient by doing on of the following.

$$f(x)\,g(x)=\frac{g(x)}{1/f(x)} \qquad \text{OR} \qquad f(x)\,g(x)=\frac{f(x)}{1/g(x)}$$

Let's take a look at another example.

Example 3 Evaluate the following limit.

$$\lim_{x\to -\infty} x\mathbf{e}^x$$

Solution

So, it's in the form $(\infty)(0)$. This means that we'll need to write it as a quotient. Moving the *x* to the denominator worked in the previous example so let's try that with this problem as well.

$$\lim_{x\to-\infty} x\mathbf{e}^x = \lim_{x\to-\infty} \frac{\mathbf{e}^x}{1/x} = \lim_{x\to-\infty} \frac{\mathbf{e}^x}{-1/x^2} = \lim_{x\to-\infty} \frac{\mathbf{e}^x}{2/x^3} = \lim_{x\to-\infty} \frac{\mathbf{e}^x}{-6/x^4} = \cdots$$

Hummmm…. This doesn't seem to be getting us anywhere. With each application of L'Hospital's Rule we just end up with another 0/0 indeterminate form.

This means that we moved the wrong function to the denominator. Let's move the exponential function instead.

$$\lim_{x\to-\infty} x\mathbf{e}^x = \lim_{x\to-\infty} \frac{x}{1/\mathbf{e}^x} = \lim_{x\to-\infty} \frac{x}{\mathbf{e}^{-x}}$$

Note that we used the fact that,

$$\frac{1}{\mathbf{e}^x} = \mathbf{e}^{-x}$$

to simplify the quotient up a little. This is now an indeterminate form of $-\infty/\infty$.

$$\lim_{x\to-\infty} x\mathbf{e}^x = \lim_{x\to-\infty} \frac{x}{\mathbf{e}^{-x}} = \lim_{x\to-\infty} \frac{1}{-\mathbf{e}^{-x}} = 0$$

So, when faced with a product $(0)(\pm\infty)$ we can turn it into a quotient that will allow us to use L'Hospital's Rule. However, as we saw in the last example we need to be careful with how we do that on occasion. Sometimes we can use either quotient and in other cases only one will work.

Let's now take a look at the indeterminate forms,

$$1^\infty \qquad 0^0 \qquad \infty^0$$

These can all be dealt with in the following way.

Example 4 Evaluate the following limit.

$$\lim_{x\to\infty} x^{\frac{1}{x}}$$

Solution

In the limit this is the indeterminate form ∞^0. We're actually going to spend most of this problem on a different limit. Let's first define the following.

$$y = x^{\frac{1}{x}}$$

Now, if we take the natural log of both sides we get,

$$\ln(y) = \ln\left(x^{\frac{1}{x}}\right) = \frac{1}{x}\ln x = \frac{\ln x}{x}$$

Let's now take a look at the following limit.

$$\lim_{x\to\infty}\ln\left(y\right)=\lim_{x\to\infty}\frac{\ln x}{x}=\lim_{x\to\infty}\frac{1/x}{1}=0$$

This limit was just a L'Hospital's Rule. So, what did this have to do with our limit? Well first notice that,

$$\mathbf{e}^{\ln(y)}=y$$

and so our limit could be written as,

$$\lim_{x\to\infty}x^{\frac{1}{x}}=\lim_{x\to\infty}y=\lim_{x\to\infty}\mathbf{e}^{\ln(y)}$$

We can now use the limit above to finish this problem.

$$\lim_{x\to\infty}x^{\frac{1}{x}}=\lim_{x\to\infty}y=\lim_{x\to\infty}\mathbf{e}^{\ln(y)}=\mathbf{e}^{\lim\limits_{x\to\infty}\ln(y)}=\mathbf{e}^{0}=1$$

With L'Hospital's Rule we are now able to take the limit of a wide variety of indeterminate forms that we were unable to deal with prior to this section.

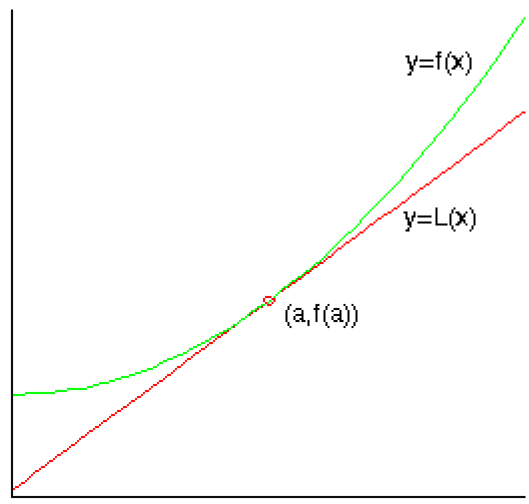
Linear Approximations

In this section we're going to take a look at an application not of derivatives but of the tangent line to a function. Of course, to get the tangent line we do need to take derivatives, so in some way this is an application of derivatives as well.

Given a function *f(x)* we can find its tangent at *x=a*.

$$L\left(x\right)=f\left(a\right)+f'\left(a\right)\left(x-a\right)$$

Take a look at the following graph of a function and its tangent line.



From this graph we can see that near *x=a* the tangent line and the function have nearly the same graph. Sometimes we will use the tangent line, *L(x)*, as an approximation to the function, *f(x)*, near *x=a*. In these cases we call the tangent line the **linear approximation** to the function at *x=a*.

So, why do would we do this? Let's take a look at an example.

Example 1 Determine the linear approximation for $f(x)=\sqrt[3]{x}$ at *x=8*. Use the linear approximation to approximate the value of $\sqrt[3]{8.05}$ and $\sqrt[3]{25}$.

Solution
Since this is just the tangent line there really isn't a whole lot to finding the linear approximation.

$$f'(x)=\frac{1}{3}x^{-\frac{2}{3}}=\frac{1}{3\sqrt[3]{x^2}} \qquad f(8)=2 \qquad f'(8)=\frac{1}{12}$$

The linear approximation is then,

$$L(x)=2+\frac{1}{12}(x-8)=\frac{1}{12}x+\frac{4}{3}$$

Now, the approximations are nothing more than plugging these into the linear approximation.

$$L(8.05)=2.00416667 \qquad \sqrt[3]{8.05}=2.00415802$$
$$L(25)=3.41666667 \qquad \sqrt[3]{25}=2.92401774$$

So, at *x=8.05* this linear approximation does a very good job of approximating the actual value. However, at *x=25* it doesn't do such a good job.

Linear approximations do a very good job of approximating values of *f(x)* “near” *x=a*. However, the farther away from *x=a* we get the worse the approximation is liable to be. Note as well that how near we need to stay to *x=a* for a good approximation will depend upon the function and *x=a* that we’re using.

Example 2 Determine the linear approximation for $\sin\theta$ at $\theta = 0$

Solution
Again, there really isn’t a whole lot to this example.

$$f(\theta) = \sin\theta \qquad f'(\theta) = \cos\theta$$
$$f(0) = 0 \qquad f'(0) = 1$$

$$\begin{aligned} L(\theta) &= f(0) + f'(\theta)(\theta - a) \\ &= 0 + (1)(\theta - 0) \\ &= \theta \end{aligned}$$

So, as long as θ is small we can say that $\sin\theta \approx \theta$

This is actually a somewhat important linear approximation. In optics this linear approximation is often used to simplify formulas. This linear approximation is also used to help describe the motion of a pendulum and vibrations in a string.

Differentials

In this section we’re going to introduce a notation that we’ll be seeing quite a bit in the next chapter. We will also look at an application of this new notation.

Given a function *y=f(x)* we call *dx* and *dy* differentials and the relationship between them is given by,

$$dy = f'(x)\,dx$$

There is a nice application to differentials. If we think of Δx as the change in *x* then $\Delta y = f(x + \Delta x) - f(x)$ is the change in *y* corresponding to the change in *x*. Now, if Δx is small we can assume that $\Delta y \approx dy$.

Example 1 Compute *dy* and Δy if $y = \cos(x^2 + 1) - x$ as *x* changes from *x=2.0* to *x=2.03*.

Solution
First let’s compute the change in *y*.

$$\Delta y = \cos\left(\left(2.03\right)^2 + 1\right) - 2.03 - \left(\cos\left(2^2 + 1\right) - 2\right) = 0.083581127$$

Now let's get the formula for *dy*.

$$dy = \left(-2x\sin\left(x^2 + 1\right) - 1\right)dx$$

Next if we have *x=2* and we assume that $dx = \Delta x = 0.03$ then,

$$dy = \left(-2\left(2\right)\sin\left(2^2 + 1\right) - 1\right)\left(0.03\right) = 0.085070913$$

So, we can see that in fact $\Delta y \approx dy$ provided we keep Δx small. We can use this fact in the following way.

Example 2 A sphere was measured and its radius was found to be 45 inches with a possible error of no more that 0.01 inches. What is the maximum possible error in the volume if we use this value of the radius?

Solution

First, recall the equation for the volume of a sphere.

$$V = \frac{4}{3}\pi r^3$$

Now, if we start with *r=45* and use $\Delta r = dr = 0.01$ then $\Delta V \approx dV$ should give us maximum error.

So, first get the formula for the differential.

$$dV = 4\pi r^2 dr$$

Now compute *dV*.

$$\Delta V \approx dV = 4\pi\left(45\right)^2\left(0.01\right) = 254.46 \text{ in}^3$$

The maximum error in the volume is then 254.46 in^3.

Be careful to not assume this is a large error. On the surface it looks large, however if we compute the actual volume for *r=45* we get *V*=381703.51 in^3. So, in comparison this isn't all that large!

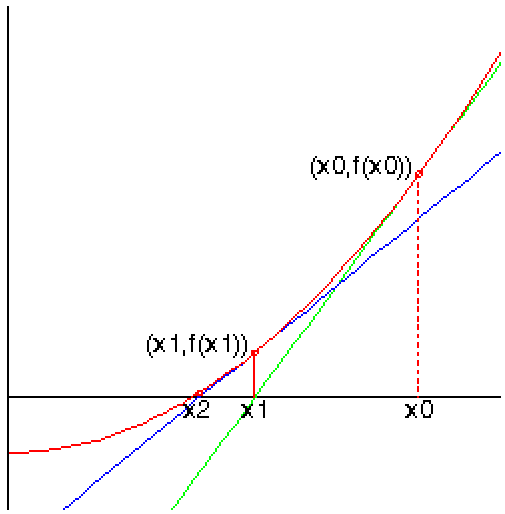
Newton's Method

The last application that we'll take a look at in this chapter is a method for approximating solutions to equations.

Let's suppose that we want to approximate the solution to *f(x)=0* and that we have somehow found an initial approximation to this solution say, x_0. This initial approximation is probably not all that good and so we'd like to find a better approximation. This is easy enough to do. First we will get the tangent line to *f(x)* at x_0.

$$y = f\left(x_0\right) + f'\left(x_0\right)\left(x - x_0\right)$$

Take a look at the graph below.



The green line is the tangent line that we just found. We can see that this line will cross the *x*-axis much closer to the actual solution to the equation than x_0. Let's call this point x_1 and we'll use this point as our new approximation to the solution.

So, how do we find this point? Well we know it's coordinates, (x_1,0), and we know that it's on the tangent line so plug this point into the tangent line and solve for x_1.

$$\begin{aligned} 0 &= f\left(x_0\right) + f'\left(x_0\right)\left(x_1 - x_0\right) \\ x_1 - x_0 &= -\frac{f\left(x_0\right)}{f'\left(x_0\right)} \\ x_1 &= x_0 - \frac{f\left(x_0\right)}{f'\left(x_0\right)} \end{aligned}$$

So, we can find the new approximation provided the derivative isn't zero at the original approximation.

Now we repeat the whole process to find an even better approximation. We form up the tangent line to *f(x)* at x_1 and use its root, x_2, as a new approximation to the actual solution. If we do this we will arrive at the following formula.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

As we can see from the graph above that following this process will get a sequence of numbers that are getting very close the actual solution. This process is called Newton's Method.

Here is the general Newton's Method

Newton's Method

If x_n is an approximation to $f(x)=0$, if $f'(x_n) \neq 0$ the next approximation is given by,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This should lead to the question of when do we stop? How many times do we go through this process? In general we continue through the process until two successive approximations agree to a given number of decimal places.

Let's work an example of Newton's Method.

Example 1 Use Newton's Method to determine an approximation to the solution to $\cos x = x$ that lies in the interval [0,2] to six decimal places.

Solution

First note that we weren't given an initial guess. We were however, given an interval in which to look. We will use this to get our initial guess. The general rule of thumb in these cases is to take the initial approximation to be the midpoint of the interval. So, we'll use $x_0=1$ as our initial guess.

Now we need the general formula for Newton's Method. First notice that we **must** have the function in the form $f(x)=0$. Therefore, we first rewrite the equation as,

$$\cos x - x = 0$$

We can now write down the general formula for Newton's Method.

$$x_{n+1} = x_n - \frac{\cos x - x}{-\sin x - 1}$$

Let's now get the first approximation.

$$x_1 = 1 - \frac{\cos(1) - 1}{-\sin(1) - 1} = 0.7503638679$$

At this point we should point out that the phrase "six decimal places" does not mean just get x_1 to six decimal places and then stop. Instead it means that we continue until two successive approximations agree to six decimal places.

Given that stopping condition we clearly need to go at least one step farther.

$$x_2 = 0.7503638679 - \frac{\cos(0.7503638679) - 0.7503638679}{-\sin(0.7503638679) - 1} = 0.7391128909$$

Alright, we're making progress. We've got the approximation to 1 decimal place. Let's do another one, leaving the details of the computation to you.

$$x_3 = 0.7390851334$$

We've got it to three decimal places. We'll need another one.

$$x_4 = 0.7390851332$$

And now we've got two approximations that agree to 9 decimal places and so we can stop. We will assume that the solution is approximately $x_4 = 0.7390851332$.

In this last example we saw that we didn't have to do too many computations in order for Newton's Method to give us an approximation in the desired range of accuracy. This will not always be the case. Sometimes it will take many iterations through the process to get to the desired accuracy and on occasion it can fail completely.

The following example is a little silly but it makes the point about the method failing.

Example 2 Use $x_0=1$ to find the approximation to the solution to $\sqrt[3]{x} = 0$

Solution

Yes, it's a silly example, but it does make the point. Let's get the general formula for Newton's method.

$$x_{n+1} = x_n - \frac{x_n^{\frac{1}{3}}}{\frac{1}{3} x_n^{-\frac{2}{3}}} = x_n - 3x_n = -2x_n$$

In fact we don't really need to do any computations here. Clearly the solution is $x=0$ however, this will get farther and farther away from the solution with each iteration. Here are a couple of computations to make the point.

$$\begin{aligned} x_1 &= -2 \\ x_2 &= 4 \\ x_3 &= -8 \\ x_4 &= 16 \\ &\textit{etc.} \end{aligned}$$

So, we need to be a little careful with Newton's method. It will usually quickly find an approximation to an equation. However, there are times when it will take a lot of work or when it won't work at all.

Integrals

Introduction

In this chapter we will be looking at integrals. Integrals are the third and final major topic that will be covered in this class. As with derivatives this chapter will be devoted almost exclusively to finding and computing integrals. Applications of integrals will be given in the following chapter.

There are really two types of integrals that we'll be looking at in this chapter : Indefinite Integrals and Definite Integrals. The first half of this chapter is devoted to indefinite integrals and the last half is devoted to definite integrals. As we will see in the last half of the chapter if we don't know indefinite integrals we will not be able to do definite integrals.

Here is a quick listing of the material that is in this chapter.

Indefinite Integrals – In this section we will start with the definition of indefinite integral. This section will be devoted mostly to the definition and properties of indefinite integrals.

Computing Indefinite Integrals – In this section we will compute some indefinite integrals and take a look at a quick application of indefinite integrals.

Substitution Rule for Indefinite Integrals – Here we will look at the Substitution Rule as it applies to indefinite integrals. Many of the integrals that we'll be doing later on in the course and in later courses will require use of the substitution rule.

More Substitution Rule – Even more substitution rule problems. The substitution rule problems were split into two sections for presentation on the web. This kept the page from getting too large.

Area Problem – In this section we start off with the motivation for definite integrals and give one of the interpretations of definite integrals.

Definition of the Definite Integral – We will formally define the definite integral in this section and give many of its properties. We will also take a look at the first part of the Fundamental Theorem of Calculus.

Computing Definite Integrals – We will take a look at the second part of the Fundamental Theorem of Calculus in this section and start to compute definite integrals.

Substitution Rule for Definite Integrals – In this section we will revisit the substitution rule as it applies to definite integrals.

Indefinite Integrals

In the past two chapters we've been given a function *f(x)* and asking what the derivative of this function was. We now want to turn things around. We now want to ask what function we differentiated to get the function *f(x)*.

Let's take a quick look at an example to get us started.

Example 1 What function did we differentiate to get the following function.

$$f(x) = x^4 + 3x - 9$$

Solution

Let's actually start by getting the derivative of this function.

$$f'(x) = 4x^3 + 3$$

The point of this was to remind us of how differentiation works. When differentiating powers of x we multiply the term by the original exponent and then drop the exponent by one.

Now, let's go back and work the problem. In fact let's just start with the first term. We got x^4 by differentiating a function and since we drop the exponent by one it looks like we must have differentiated x^5. However, if we had differentiated x^5 we would have $5x^4$ and we don't have a 5 in front our first term, so the 5 needs to cancel out after we've differentiated. It looks then like we would have to differentiate $\frac{1}{5}x^5$ in order to get x^4.

Likewise for the second term, in order to get $3x$ after differentiating we would have to differentiate $\frac{3}{2}x^2$. Again, the fraction is there to cancel out the *2* we pick up in the differentiation.

The third term is just a constant and we know that if we differentiate x we get 1. So, it looks like we had to differentiate -9x to get the last term.

Putting all of this together gives the following function,

$$F(x) = \frac{1}{5}x^5 + \frac{3}{2}x^2 - 9x$$

Our answer is easy enough to check. Simply differentiate *F(x)*.

$$F'(x) = x^4 + 3x - 9 = f(x)$$

So, it looks like we got the correct function. Or did we? We know that the derivative of a constant is zero and so any of the following will also give *f(x)* upon differentiating.

$$\begin{aligned} &F\left(x\right)=\frac{1}{5}x^5+\frac{3}{2}x^2-9x+10\\ &F\left(x\right)=\frac{1}{5}x^5+\frac{3}{2}x^2-9x-1954\\ &F\left(x\right)=\frac{1}{5}x^5+\frac{3}{2}x^2-9x+\frac{3469}{123}\\ &etc.\end{aligned}$$

In fact, any function of the form,

$$F\left(x\right)=\frac{1}{5}x^5+\frac{3}{2}x^2-9x+c, \qquad c \text{ is a constant}$$

will give *f(x)* upon differentiating.

There were two points to this last example. The first point was to get you thinking about how to do these problems. It is important initially to remember that we are really just asking what we differentiated to get the given function.

The other point is to recognize that there are actually an infinite number of functions that we could use and they will all differ by a constant.

Now that we've worked an example let's get some of the definitions and terminology out of the way.

Definitions

Given a function *f(x)* an **anti-derivative** of *f(x)* is any function *F(x)* such that

$$F'\left(x\right)=f\left(x\right)$$

If *F(x)* is any anti-derivative of *f(x)* then the most general anti-derivative of *f(x)* is called an **indefinite integral** and denoted,

$$\int f\left(x\right)dx=F\left(x\right)+c, \qquad c \text{ is any constant}$$

In this definition the $\int$ is called the **integral symbol**, *f(x)* is called the **integrand**, *x* is called the **integration variable** and the "*c*" is called the **constant of integration**.

Note that often we will just say integral instead of indefinite integral (or definite integral for that matter when we get to those). It will be clear from the context of the problem that we are talking about an indefinite integral (or definite integral).

The process of finding the indefinite integral is called **integration** or **integrating** ***f(x)***. If we need to be specific about the integration variable we will say that we are **integrating *f(x)* with respect to *x***.

Let's rework the first problem in light of the new terminology.

Example 2 Evaluate the following indefinite integral.

$$\int x^4 + 3x - 9\,dx$$

Solution

Since this is really asking for the most general anti-derivative we just need to reuse the final answer from the first example.

The indefinite integral is,

$$\int x^4 + 3x - 9\,dx = \frac{1}{5}x^5 + \frac{3}{2}x^2 - 9x + c$$

A couple of warnings are now in order. One of the more common mistakes that students make with integrals (both indefinite and definite) is to drop the *dx* at the end of the integral. This is required! Think of the integral sign and the *dx* as a set of parenthesis. Every time you open a parenthesis you must close it. With integrals, think of the integral sign as an "open parenthesis" and the *dx* as a "close parenthesis".

If you drop the *dx* it won't be clear where the integrand ends. Consider the following variations of the above example.

$$\int x^4 + 3x - 9\,dx = \frac{1}{5}x^5 + \frac{3}{2}x^2 - 9x + c$$

$$\int x^4 + 3x\,dx - 9 = \frac{1}{5}x^5 + \frac{3}{2}x^2 + c - 9$$

$$\int x^4\,dx + 3x - 9 = \frac{1}{5}x^5 + c + 3x - 9$$

You only integrate what is between the integral sign and the *dx*. Each of the above integrals end in a different place and so we get different answers because we integrate a different number of terms each time. In the second integral the "-9" is outside the integral and so is left alone and not integrated. Likewise, in the third integral the "3*x*-9" is outside the integral and so is left alone.

Knowing which terms to integrate is not the only reason for writing the *dx* down. In the Substitution Rule section we will actually be working with the *dx* and if we aren't in the habit of writing it down we will get the wrong answer at that stage.

The moral of this is to make sure and put in the *dx*! At this stage it may seem like a silly thing to do, but it just needs to be there, if for no other reason than knowing where the integral stops.

On a side note, the *dx* notation should seem a little familiar to you. We saw things like this a couple of sections ago. We called the *dx* a differential in that section and yes that is exactly what it is. The *dx* that ends the integral is nothing more than a differential.

The next topic that we should discuss here is the integration variable used in the integral. Actually there isn't really a lot to discuss here other than to note that the integration variable doesn't really matter. For instance,

$$\int x^4 + 3x - 9\,dx = \frac{1}{5}x^5 + \frac{3}{2}x^2 - 9x + c$$
$$\int t^4 + 3t - 9\,dt = \frac{1}{5}t^5 + \frac{3}{2}t^2 - 9t + c$$
$$\int w^4 + 3w - 9\,dw = \frac{1}{5}w^5 + \frac{3}{2}w^2 - 9w + c$$

Changing the integration variable in the integral simply changes the variable in the answer. It is important to notice however that when we change the integration variable in the integral we also changed the differential to match the new variable. This is more important that we might realize at this point.

Another use of the differential at the end of integral is to tell us what variable we are integrating with respect to. At this stage that may seem unimportant since most of the integrals that we're going to be working with here will only involve a single variable. However, if you are on a degree track that will take you into multi-variable calculus this will be very important at that stage since there will be more than one variable in the problem. You need to get into the habit of writing the correct differential at the end of the integral so when it becomes important you will already be in the habit of writing it down.

To see why this is important take a look at the following two integrals.

$$\int 2x\,dx$$
$$\int 2t\,dx$$

The first integral is simple enough.

$$\int 2x\,dx = x^2 + c$$

The second integral is also fairly simple, but we need to be careful. The *dx* tells us that we are integrating *x*'s. That means that we only integrate *x*'s that are in the integrand and all other variables in the integrand are considered to be constants. The second integral is then,

$$\int 2t\,dx = 2tx + c$$

So, it may seem silly to always put in the *dx*, but it is a vital bit of notation that can cause us to get the incorrect answer if we neglect to put it in.

Now, there are some important properties of integrals that we should take a look at.

Properties of the Indefinite Integral

1. $\int cf(x)\,dx = c\int f(x)\,dx$ where *c* is any number. So, we can factor multiplicative constants out of indefinite integrals.

2. $\int -f(x)\,dx = -\int f(x)\,dx$. This is really the first property with *c=-1*.

3. $\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$. In other words, the integral of a sum or difference of functions is the sum or difference of the individual integrals. This rule can be extended to as many functions as we need.

Notice that when we worked the first example we used the first and third property. We integrated each term individually, put any constants back in and the put everything back together with the appropriate sign.

Not listed in the properties above were integrals of products and quotients. The reason for this is simple. Just like with derivatives each of the following will NOT work.

$$\int f(x)\,g(x)\,dx \neq \int f(x)\,dx \int g(x)\,dx$$
$$\int \frac{f(x)}{g(x)}\,dx \neq \frac{\int f(x)\,dx}{\int g(x)\,dx}$$

With derivatives we had a product rule and a quotient rule to deal with these cases. However, with integrals there are no such rules. When faced with a product and quotient in an integral we will have a variety of ways of dealing with it depending on just what the integrand is.

There is one final topic to be discussed briefly in this section. On occasion we will be given $f'(x)$ and will ask what *f(x)* was. We can now answer this with an indefinite integral.

$$f(x) = \int f'(x)\,dx$$

In this section we only evaluated a single indefinite integral. The point of this section was not to do indefinite integrals, but instead to get us familiar with the notation and some of the basic ideas and properties of indefinite integrals. The next couple of sections are devoted to actually evaluating indefinite integrals.

Computing Indefinite Integrals

In the previous section we started looking at indefinite integrals and in that section we concentrated almost exclusively on notation, concepts and properties of the indefinite integral.

In this section we need to start thinking about how we actually compute indefinite integrals. We'll start off with some of the basic indefinite integrals.

The first integral that we'll look at is the integral of a power of *x*.

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c, \quad n \neq -1$$

It is clear (hopefully) that we will need to avoid *n=-1* in this formula. If we allow *n=-1* in this formula we will end up with division by zero. We will take care of this case in a bit. When integrating a power of *x* we add one onto the exponent and then divide by the new exponent.

Next is one of the easier integrals but always seems to cause problems for people.

$$\int k\,dx = kx + c, \qquad c \text{ and } k \text{ are constants}$$

Remember that all we're asking is what did we differentiate to get the integrand.

Let's now take a look at the trig functions.

$$\int \sin x\,dx = -\cos x + c \qquad\qquad \int \cos x\,dx = \sin x + c$$

$$\int \sec^2 x\,dx = \tan x + c \qquad\qquad \int \sec x \tan x\,dx = \sec x + c$$

$$\int \csc^2 x\,dx = -\cot x + c \qquad\qquad \int \csc x \cot x\,dx = -\csc x + c$$

Notice that we only integrated two of the six trig functions here. The remaining four integrals are really integrals that give the remaining four trig functions. Also, be careful with signs here. It is easy to get the signs for derivatives and integrals mixed up. Again, remember that we're asking what function we differentiated to get the integrand.

We will be able to integrate the remaining four trig functions in a couple of sections.

Now, let's take care of exponential and logarithm functions.

$$\int \mathbf{e}^x\,dx = \mathbf{e}^x + c \qquad \int a^x\,dx = \frac{a^x}{\ln a} + c \qquad \int \frac{1}{x}\,dx = \int x^{-1}\,dx = \ln|x| + c$$

Integrating logarithms requires a topic that is usually taught in Calculus II and so we won't be integrating a logarithm in this class. Also note the third integrand can be written in a couple of ways and don't forget the absolute value bars in the *x* in the answer to the third integral.

Finally, let's take care of the inverse and hyperbolic trig functions.

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1} x + c \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + c$$

$$\int \sinh x\,dx = \cosh x + c \qquad \int \cosh x\,dx = \sinh x + c$$

$$\int \operatorname{sech}^2 x\,dx = \tanh x + c \qquad \int \operatorname{sech} x \tanh x\,dx = -\operatorname{sech} x + c$$

$$\int \operatorname{csch}^2 x\,dx = -\coth x + c \qquad \int \operatorname{csch} x \coth x\,dx = -\operatorname{csch} x + c$$

As with logarithms integrating inverse trig functions requires a topic usually taught in Calculus II and so we won't be integrating them in this class. There is also a different answer for the second integral above. Recalling that since all we are asking here is what function did we differentiate to get the integrand the second integral could also be,

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = -\cos^{-1} x + c$$

Traditionally we use the first form of this integral.

Okay, now that we've got most of the basic integrals out of the way let's do some indefinite integrals. In all of these problems remember that we can always check our answer by differentiating and making sure that we get the integrand.

Example 1 Evaluate each of the following indefinite integrals.

(a) $\int 5t^3 - 10t^{-6} + 4\,dt$

(b) $\int x^8 + x^{-8}\,dx$

(c) $\int 3\sqrt[4]{x^3} + \frac{7}{x^5} + \frac{1}{6\sqrt{x}}\,dx$

(d) $\int dy$

(e) $\int \left(w + \sqrt[3]{w}\right)\left(4 - w^2\right)dw$

(f) $\int \frac{4x^{10} - 2x^4 + 15x^2}{x^3}\,dx$

Solution

Okay, in all of these remember the basic rules of indefinite integrals. First, to integrate sums and differences all we really do is integrate the individual terms and then put them back together with the appropriate signs. Next, we can ignore any coefficients until we are done with integrating that particular term and then put the coefficient back in. Also, do not forget the "+*c*" at the end it is important and must be there.

So, let's evaluate some integrals.

(a) There's not really a whole lot to do here other than use the first two formulas from the beginning of this section.

$$\int 5t^3 - 10t^{-6} + 4\,dt = 5\left(\frac{1}{4}\right)t^4 - 10\left(\frac{1}{-5}\right)t^{-5} + 4t + c$$
$$= \frac{5}{4}t^4 + 2t^{-5} + 4t + c$$

Be careful when integrating negative exponents. Remember to add one onto the exponent. One of the more common mistakes that students make when integrating negative exponents is to "add one" and end up with an exponent of "-7" instead of the correct exponent of "-5".

(b) This is here just to make sure we get the point about integrating negative exponents.

$$\int x^8 + x^{-8}\,dx = \frac{1}{9}x^9 - \frac{1}{7}x^{-7} + c$$

(c) In this case there isn't a formula for explicitly dealing with radicals or rational expressions. However, just like with derivatives we can write all these terms so they are in the numerator and they all have an exponent. This should always be your first step when faced with this kind of integral.

$$\int 3\sqrt[4]{x^3} + \frac{7}{x^5} + \frac{1}{6\sqrt{x}}\,dx = \int 3x^{\frac{3}{4}} + 7x^{-5} + \frac{1}{6}x^{-\frac{1}{2}}\,dx$$
$$= 3\frac{1}{\frac{7}{4}}x^{\frac{7}{4}} - \frac{7}{4}x^{-4} + \frac{1}{6}\left(\frac{1}{\frac{1}{2}}\right)x^{\frac{1}{2}} + c$$
$$= \frac{12}{7}x^{\frac{7}{4}} - \frac{7}{4}x^{-4} + \frac{1}{3}x^{\frac{1}{2}} + c$$

When dealing with fractional exponents we usually don't "divide by the new exponent". Doing this is equivalent to multiplying the by the reciprocal of the new exponent and so that is what we will usually do.

(d) Don't make this one harder that it is…

$$\int dy = \int 1\,dy = y + c$$

In this case we are really just integrating a one!

(e) We've got a product and as we noted in the previous section there is no rule for dealing with products. However, in this case we don't need a rule. All that we need to do is multiply things out and then we will be able to integrate.

$$\int \left(w + \sqrt[3]{w}\right)\left(4 - w^2\right)dw = \int 4w - w^3 + 4w^{\frac{1}{3}} - w^{\frac{7}{3}}dw$$
$$= 2w^2 - \frac{1}{4}w^4 + 3w^{\frac{4}{3}} - \frac{3}{10}w^{\frac{10}{3}} + c$$

(f) As with the previous problem it's not really a problem that we don't have a rule for quotients. In this case all we need to do is break up the quotient and then integrate the individual terms.

$$\begin{aligned}\int \frac{4x^{10}-2x^4+15x^2}{x^3}\,dx &= \int \frac{4x^{10}}{x^3}-\frac{2x^4}{x^3}+\frac{15x^2}{x^3}\,dx \\ &= \int 4x^7-2x+\frac{15}{x}\,dx \\ &= \frac{1}{2}x^8-x^2+15\ln|x|+c\end{aligned}$$

Be careful to not think of the third term as x to a power. Using that rule on the third term will NOT work. The third term is simply a logarithm. Also, don't get excited about the 15. Recall that the 15 is just a constant and so it can be factored out of the integral. In other words, here is what we did to integrate the third term.

$$\int \frac{15}{x}\,dx = 15\int \frac{1}{x}\,dx = 15\ln|x|+c$$

Always remember that you can't integrate products and quotients in the same way that we integrate sums and differences. At this point the only way to integrate products and quotients is to multiply the product or break up the quotient.

The first set of examples focused almost exclusively on powers of x (or whatever variable we used in each example). It's time to do some examples that involve other functions.

Example 2 Evaluate each of the following integrals.

(a) $\int 3\mathbf{e}^x+5\cos x-10\sec^2 x\,dx$

(b) $\int 2\sec w\tan w+\frac{1}{6w}\,dw$

(c) $\int \frac{23}{y^2+1}+6\csc y\cot y+\frac{9}{y}\,dy$

(d) $\int \frac{3}{\sqrt{1-x^2}}+6\sin x+10\sinh x\,dx$

(e) $\int \frac{7-6\sin^2\theta}{\sin^2\theta}\,d\theta$

(f) $\int \sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)dt$

Solution

Most of the problems in this example will simply use the formulas from the beginning of this section.

(a) There isn't anything to this one other than using the formulas.

$$\int 3\mathbf{e}^x + 5\cos x - 10\sec^2 x\,dx = 3\mathbf{e}^x + 5\sin x - 10\tan x + c$$

(b) Let's be a little careful with this one. First break it up into two integrals and note the rewritten integrand on the second integral.

$$\begin{aligned}\int 2\sec w\tan w + \frac{1}{6w}\,dw &= \int 2\sec w\tan w\,dw + \int \frac{1}{6}\frac{1}{w}\,dw \\ &= \int 2\sec w\tan w\,dw + \frac{1}{6}\int \frac{1}{w}\,dw\end{aligned}$$

Having rewrote the second integrand will help with the integration. We can think of the 6 in the denominator as a 1/6 out in front of the term and then since this is a constant it can be factored out of the integral. The answer is then,

$$\int 2\sec w\tan w + \frac{1}{6w}\,dw = 2\sec w + \frac{1}{6}\ln|w| + c$$

(c) In this one we'll just use the formulas from above and don't get excited about the coefficients.

$$\int \frac{23}{y^2+1} + 6\csc y\cot y + \frac{9}{y}\,dy = 23\tan^{-1} y - 6\csc y + 9\ln|y| + c$$

(d) Again, there really isn't a whole lot to do with this one other than to use the appropriate formula from above.

$$\int \frac{3}{\sqrt{1-x^2}} + 6\sin x + 10\sinh x\,dx = 3\sin^{-1} x - 6\cos x + 10\cosh x + c$$

(e) This one can be a little tricky if you aren't ready for it. At this point the only way we have of dealing with quotients is to break it up.

$$\begin{aligned}\int \frac{7-6\sin^2\theta}{\sin^2\theta}\,d\theta &= \int \frac{7}{\sin^2\theta} - 6\,d\theta \\ &= \int 7\csc^2\theta - 6\,d\theta\end{aligned}$$

Notice that upon breaking the integral up we further simplified the integrand by recalling the definition of cosecant. We can now do the integral.

$$\int \frac{7-6\sin^2\theta}{\sin^2\theta}\,d\theta = -7\cot\theta - 6\theta + c$$

(f) There are several ways to do this integral and most of them require the next section. However, there is a way to do this integral using only the material from this section. All that is required is to remember the trig formula,

$$\sin(2t) = 2\sin t\cos t$$

and so,

$$\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right) = \frac{1}{2}\sin(t)$$

Using this formula the integral becomes,

$$\begin{aligned}\int \sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)dt &= \int \frac{1}{2}\sin(t)\,dt \\ &= -\frac{1}{2}\cos(t) + c\end{aligned}$$

As noted earlier there is another method for doing this integral. In fact there are two alternate methods. To see all three check out the section on Constant of Integration in the Extras chapter.

Again, we need to be careful with products and quotients. We can't just integrate the individual pieces and then multiply or divide the results back up.

There is one more set of examples that we should do before moving out of this section.

Example 3 Given the following information determine the function *f(x)*.

(a) $f'(x) = 4x^3 - 9 + 2\sin x + 7\mathbf{e}^x, \; f(0) = 15$

(b) $f''(x) = 15\sqrt{x} + 5x^3 + 6, \quad f(1) = -\frac{5}{4}, \; f(4) = 404$

Solution

In both of these we will need to remember that

$$f(x) = \int f'(x)\,dx$$

Also note that because we are giving values of the function at specific points we are also going to be determining what the constant of integration will be.

(a) The first step here is to integrate.

$$\begin{aligned}f(x) &= \int 4x^3 - 9 + 2\sin x + 7\mathbf{e}^x\,dx \\ &= x^4 - 9x - 2\cos x + 7\mathbf{e}^x + c\end{aligned}$$

Now we have a value of the function so let's plug in $x=0$ and determine the value of the constant of integration c.

$$\begin{aligned}15 = f(0) &= -2\cos(0) + 7\mathbf{e}^0 + c \\ &= -2 + 7 + c \\ &= 5 + c\end{aligned}$$

So, from this it looks like $c=10$. This means that the function is,

$$f(x) = x^4 - 9x - 2\cos x + 7\mathbf{e}^x + 10$$

(b) This one is a little different form the first one. In order to get the function we will need the first derivative and we have the second derivative. We can however, use an integral to get the first derivative from the second derivative, just as we used an integral to get the function from the first derivative.

$$\begin{aligned} f'(x) &= \int f''(x)\,dx \\ &= \int 15x^{\frac{1}{2}} + 5x^3 + 6\,dx \\ &= 15\left(\frac{2}{3}\right)x^{\frac{3}{2}} + \frac{5}{4}x^4 + 6x + c \\ &= 10x^{\frac{3}{2}} + \frac{5}{4}x^4 + 6x + c \end{aligned}$$

Don't forget the constant of integration!

We can now find the function.

$$\begin{aligned} f(x) &= \int 10x^{\frac{3}{2}} + \frac{5}{4}x^4 + 6x + c\,dx \\ &= 4x^{\frac{5}{2}} + \frac{1}{4}x^5 + 3x^2 + cx + d \end{aligned}$$

There will be no reason to think the constants of integration from each integration will be the same and so we'll need to call them something different. Also, don't get excited about integrating the *c*. It's just a constant and we know how to integrate constants.

Now, plug in the two values of the function that we've got.

$$-\frac{5}{4} = f(1) = 4 + \frac{1}{4} + 3 + c + d = \frac{29}{4} + c + d$$

$$404 = f(4) = 4(32) + \frac{1}{4}(1024) + 3(16) + c(4) + d = 432 + 4c + d$$

This gives us a system of two equations in two unknowns that we can solve.

$$\begin{aligned} -\frac{5}{4} &= \frac{29}{4} + c + d \\ 404 &= 432 + 4c + d \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} c &= -\frac{13}{2} \\ d &= -2 \end{aligned}$$

The function is then,

$$f(x) = 4x^{\frac{5}{2}} + \frac{1}{4}x^5 + 3x^2 - \frac{13}{2}x - 2$$

Don't remember how to solve systems? Check out the Solving Systems portion of my Algebra/Trig Review.

In this section we've started the process of integration. We've seen how to do quite a few basic integrals and we also saw a quick application of integrals in the last example.

There are many new formulas in this section that we'll now have to know. However, if you think about it, they aren't really new formulas. They are really nothing more than derivative formulas that we should already know written in terms of integrals. If you remember that you should find if easier to remember the formulas in this section.

Always remember that integration is asking nothing more than what function did we differentiate to get the integrand. If you can remember that many of the basic integrals that we saw in this section aren't too bad.

Substitution Rule for Indefinite Integrals

After the last section we now know how to do the following integrals.

$$\int \sqrt[4]{x}\,dx \qquad \int \frac{1}{t^3}\,dt \qquad \int \cos w\,dw \qquad \int \mathbf{e}^y\,dy$$

However, we can't do the following integrals.

$$\int 18x^2\sqrt[4]{6x^3+5}\,dx \qquad \int \frac{2t^3+1}{\left(t^4+2t\right)^3}\,dt$$

$$\int \left(1-\frac{1}{w}\right)\cos\left(w-\ln w\right)dw \qquad \int \left(8y-1\right)\mathbf{e}^{4y^2-y}\,dy$$

All of these look considerably more difficult than the first set. However, they aren't too bad once you see how to do them. Let's start with the first one.

$$\int 18x^2\sqrt[4]{6x^3+5}\,dx$$

In this case let's notice that if we let

$$u = 6x^3+5$$

and we compute the differential (you remember how to compute these right?) for this we get,

$$du = 18x^2 dx$$

Now, let's go back to our integral and notice that we can eliminate every x that exists in the integral and write the integral completely in terms of u using both the definition of u and its differential.

$$\begin{aligned}\int 18x^2\sqrt[4]{6x^3+5}\,dx &= \int \left(6x^3+5\right)^{\frac{1}{4}}\left(18x^2dx\right)\\ &= \int u^{\frac{1}{4}}\,du\end{aligned}$$

In the process of doing this we've taken an integral that looked very difficult and with a quick substitution we were able to rewrite the integral into a very simple integral that we can do.

Evaluating the integral gives,

$$\begin{aligned}\int 18x^2\sqrt[4]{6x^3+5}\,dx &= \frac{4}{5}u^{\frac{5}{4}}+c\\ &= \frac{4}{5}\left(6x^3+5\right)^{\frac{5}{4}}+c\end{aligned}$$

As always we can check our answer with a quick derivative if we'd like to and don't forget to "back substitute" and get the integral back into terms of the original variable.

What we've done in the work above is called the **Substitution Rule**. Here is the substitution rule in general.

Substitution Rule

$$\int f\left(g\left(x\right)\right)g'\left(x\right)dx = \int f\left(u\right)du, \qquad \text{where, } u = g\left(x\right)$$

A natural question at this stage is how to identify the correct substitution. Unfortunately, the answer is it depends on the integral. However, there is a general rule of thumb that will work for many of the integrals that we're going to be running across.

When faced with an integral we'll ask ourselves what we know how to integrate. With the integral above we can quickly recognize that we know how to integrate

$$\int \sqrt[4]{x}\,dx$$

However, we didn't have just the root we also had stuff in front of the root and (more importantly in this case) stuff under the root. Since we can only integrate roots if there is just an x under the root a good first guess for the substitution is then to make u be the stuff under the root.

How, do we know if we got the correct substitution? Well, upon computing the differential and actually performing the substitution every x in the integral (including the x in the dx) must disappear in the substitution process and the only letters left should be u's (including a du).

In our case, upon computing the differential, we can see that the stuff that is in front of the root appears in the differential and so every x will disappear in the substitution process. This will mean that we guessed correctly.

Note that often the differential will not appear exactly in the integrand and sometimes we'll need to do some manipulation of the integrand and/or the differential to get all the x's to disappear in the substitution.

Let's work some examples so we can get a better idea on how the substitution rule works.

Example 1 Evaluate each of the following integrals.

(a) $\displaystyle\int \left(1-\frac{1}{w}\right)\cos\left(w-\ln w\right)dw$

(b) $\displaystyle\int 3\left(8y-1\right)\mathbf{e}^{4y^2-y}\,dy$

(c) $\displaystyle\int x^2\left(3-10x^3\right)^4\,dx$

(d) $\displaystyle\int \cos\left(3z\right)\sin^{10}\left(3z\right)dz$

(e) $\displaystyle\int \frac{x}{\sqrt{1-4x^2}}\,dx$

Solution

(a) In this case we know how to integrate just a cosine so let's make the substitution the stuff that is inside the cosine.

$$u = w-\ln w \qquad\qquad du = \left(1-\frac{1}{w}\right)dw$$

So, as with the first example we worked the stuff in front of the cosine appears exactly in the differential. The integral is then,

$$\begin{aligned}\int \left(1-\frac{1}{w}\right)\cos\left(w-\ln w\right)dw &= \int \cos\left(u\right)du\\ &= \sin\left(u\right)+c\\ &= \sin\left(w-\ln w\right)+c\end{aligned}$$

Don't forget to go back to the original variable in the problem.

(b) Again, we know how to integrate an exponential by itself so it looks like the substitution for this problem should be,

$$u = 4y^2-y \qquad\qquad du = \left(8y-1\right)dy$$

Now, with the exception of the 3 the stuff in front of the exponential appears exactly in the differential. Recall however that we can factor the 3 out of the integral and so it won't cause any problems. The integral is then,

$$\begin{aligned}\int 3\left(8y-1\right)\mathbf{e}^{4y^2-y}\,dy &= 3\int \mathbf{e}^{u}\,du\\ &= 3\mathbf{e}^{u}+c\\ &= 3\mathbf{e}^{4y^2-y}+c\end{aligned}$$

(c) In this case it looks like the following should be the substitution.

$$u = 3-10x^3 \qquad\qquad du = -30x^2\,dx$$

Okay, now we have a small problem. We've got an x^2 out in front of the parenthesis but we don't have a "-30". This is not really the problem it might appear to be at first. We will simply rewrite the differential as follows.

$$x^2\,dx = -\frac{1}{30}\,du$$

With this we can now substitute the "$x^2\ dx$" away. In the process we will pick up a constant, but that isn't a problem since it can always be factored out of the integral.

We can now do the integral.

$$\begin{aligned}\int x^2\left(3-10x^3\right)^4 dx &= \int \left(3-10x^3\right)^4 x^2\,dx\\ &= \int u^4\left(-\frac{1}{30}\right)du\\ &= -\frac{1}{30}\left(\frac{1}{5}\right)u^5 + c\\ &= -\frac{1}{150}\left(3-10x^3\right)^5 + c\end{aligned}$$

(d) This one is a little tricky at first. We can see the correct substitution by recalling that,

$$\sin^{10}\left(3z\right) = \left(\sin\left(3z\right)\right)^{10}$$

Using this it looks like the correct substitution is,

$$u = \sin\left(3z\right) \qquad du = 3\cos\left(3z\right)dz \qquad \Rightarrow \qquad \cos\left(3z\right)dz = \frac{1}{3}du$$

Notice that we again needed to do a little manipulation of the differential to get it to work in our substitution.

Here is the integral.

$$\begin{aligned}\int \cos\left(3z\right)\sin^{10}\left(3z\right)dz &= \frac{1}{3}\int u^{10}\,du\\ &= \frac{1}{3}\left(\frac{1}{11}\right)u^{11} + c\\ &= \frac{1}{33}\sin^{11}\left(3z\right) + c\end{aligned}$$

Note that we didn't put as many steps into this one as we did in the previous example. The work in this example is more like the work that we'll typically do in these. Also note that the one third in front of the integral came about from the substitution on the differential and we just factored it out to the front of the integral.

(e) The final problem in this set. In this case the substitution is,

$$u = 1-4x^2 \qquad du = -8x\,dx \qquad \Rightarrow \qquad xdx = -\frac{1}{8}du$$

The integral is,

$$\begin{aligned}\int \frac{x}{\sqrt{1-4x^2}}\,dx &= \int x\left(1-4x^2\right)^{-\frac{1}{2}}\,dx \\ &= -\frac{1}{8}\int u^{-\frac{1}{2}}\,du \\ &= -\frac{1}{4}u^{\frac{1}{2}}+c \\ &= -\frac{1}{4}\left(1-4x^2\right)^{\frac{1}{2}}+c\end{aligned}$$

The most important thing to remember in substitution problems is that after the substitution all the original variables need to disappear in the substitution. The only variables that should be in the integral should be the new variable from the substitution (usually u). Note as well that this includes the variables in the differential!

Now let's work a set of examples that, while not particular difficult examples, can get us in trouble if we aren't paying attention.

Example 2 Evaluate each of the following integrals.

(a) $\displaystyle\int \frac{1}{\sqrt{1-4x^2}}\,dx$

(b) $\displaystyle\int \frac{3}{5y^2+4}\,dy$

(c) $\displaystyle\int \frac{2t^3+1}{\left(t^4+2t\right)^3}\,dt$

(d) $\displaystyle\int \frac{2t^3+1}{t^4+2t}\,dt$

Solution

(a) First note that this problem is very similar to the last problem in the first example set. The only difference is this one is lacking an x in the numerator. That will mean that the substitution used for that problem will not work for this problem.

Recall that after substitution all x should disappear. However, there is an x in the differential that will not disappear if we use that substitution.

The key to this integral is to recall the following formula from the previous section.

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + c$$

The integral in this problem is nearly this. The only difference is the presence of the 4. With the correct substitution this can be dealt with however. To see what this substitution should be let's rewrite the integral a little.

$$\int \frac{1}{\sqrt{1-4x^2}}\,dx = \int \frac{1}{\sqrt{1-(2x)^2}}\,dx$$

With this rewrite it looks like we can use the following substitution.

$$u = 2x \qquad du = 2dx \qquad \Rightarrow \qquad dx = \frac{1}{2}du$$

The integral is then,

$$\begin{aligned}\int \frac{1}{\sqrt{1-4x^2}}\,dx &= \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\,du \\ &= \frac{1}{2}\sin^{-1}u + c \\ &= \frac{1}{2}\sin^{-1}(2x) + c\end{aligned}$$

(b) As with the previous part this one is also an inverse trig integral. In this case the formula that we need is,

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + c$$

We will have to deal with the 5 as we did in the previous part. This time however we've got another problem. The formula that we're going to use requires the denominator to be in the form "1+…" and we've got "4+…". The first thing that we're going to have to do then is factor a 4 out of the denominator.

$$\int \frac{3}{5y^2+4}\,dy = \int \frac{3}{4\left(\frac{5y^2}{4}+1\right)}\,dy = \frac{3}{4}\int \frac{1}{\frac{5y^2}{4}+1}\,dy$$

Notice that we also factored the 3 out of the numerator. In this case the substitution will be,

$$u = \frac{\sqrt{5}y}{2} \qquad du = \frac{\sqrt{5}}{2}dy \qquad \Rightarrow \qquad dy = \frac{2}{\sqrt{5}}du$$

Don't get excited about the root in the substitution, these will show up on occasion. The integral is,

$$\int \frac{3}{5y^2+4}\,dy = \frac{3}{4}\left(\frac{2}{\sqrt{5}}\right)\int \frac{1}{u^2+1}\,du$$
$$= \frac{3}{2\sqrt{5}}\tan^{-1}\left(u\right)+c$$
$$= \frac{3}{2\sqrt{5}}\tan^{-1}\left(\frac{\sqrt{5}\,y}{2}\right)+c$$

(c) This problem could have easily been in the first set of examples. It is here mostly as a contrast to the next problem. The substitution will be,

$$u = t^4+2t \qquad du = \left(4t^3+2\right)dt = 2\left(2t^3+1\right)dt \qquad \Rightarrow \qquad \left(2t^3+1\right)dt = \frac{1}{2}du$$

Notice that in order to get this substitution to work we had to do a little more manipulation that is normally required.

The integral is,

$$\int \frac{2t^3+1}{\left(t^4+2t\right)^3}\,dt = \frac{1}{2}\int \frac{1}{u^3}\,du$$
$$= \frac{1}{2}\int u^{-3}\,du$$
$$= \frac{1}{2}\left(-\frac{1}{2}\right)u^{-2}+c$$
$$= -\frac{1}{4}\left(t^4+2t\right)^{-2}+c$$

(d) The only difference between this problem and the previous one is the denominator. In the previous problem the whole denominator is cubed and in this problem the denominator has no power on it. The same substitution will work in this problem. The differences between this problem and the previous one will come after we do the substitution.

So, using the substitution from the previous example the integral is,

$$\int \frac{2t^3+1}{t^4+2t}\,dt = \frac{1}{2}\int \frac{1}{u}\,du$$
$$= \frac{1}{2}\ln\left|u\right|+c$$
$$= \frac{1}{2}\ln\left|t^4+2t\right|+c$$

So, in this case we get a logarithm from the integral.

Since this document is also being presented on the web we're going to put the rest of the substitution rule examples in the next section. With all the examples in one section the section was becoming too large for web presentation.

More Substitution Rule

In order to allow these pages to be displayed on the web we've broken the substitution rule examples into two sections. The previous section contains the introduction to the substitution rule and some fairly basic examples. The examples in this section tend towards the slightly more difficult side.

In the first set of problems in this section the difficulty is not with the actual integration itself, but with the set up for the integration. Most of the integrals are fairly simple and most of the substitutions are fairly simple. However, some of these problems can be tricky the first time through.

Example 1 Evaluate each of the following integrals.

(a) $\int \mathbf{e}^{2t} + \sec(2t)\tan(2t)\,dt$

(b) $\int \sin(t)\left(4\cos^3(t) + 6\cos^2(t) - 8\right)dt$

(c) $\int \mathbf{e}^{-z} + \sec^2\left(\frac{z}{10}\right)dz$

(d) $\int \sin w\sqrt{1-2\cos w} + \frac{1}{7w+2}\,dw$

(e) $\int x^2 + \mathbf{e}^{1-x}\,dx$

(f) $\int \frac{10x+3}{x^2+16}\,dx$

(a) This first integral has two terms in it and both will require the same substitution.

$$u = 2t \qquad du = 2dt \qquad \Rightarrow \qquad dt = \frac{1}{2}du$$

The integral is then,

$$\begin{aligned}\int \mathbf{e}^{2t} + \sec(2t)\tan(2t)\,dt &= \frac{1}{2}\int \mathbf{e}^{u} + \sec(u)\tan(u)\,du \\ &= \frac{1}{2}\left(\mathbf{e}^{u} + \sec(u)\right) + c \\ &= \frac{1}{2}\left(\mathbf{e}^{2t} + \sec(2t)\right) + c\end{aligned}$$

Often a substitution can be used multiple times in an integral so don't get excited about that if it happens. Also note that since there was a $\frac{1}{2}$ in front of the whole integral there must also be a $\frac{1}{2}$ in front of the answer from the integral.

(b) This problem looks tricky at first. Here is the substitution for this problem,

$$u = \cos(t) \qquad du = -\sin(t)\,dt \qquad \Rightarrow \qquad \sin(t)\,dt = -du$$

In order to use this substitution there must be a sine multiplying the whole integrand as there is in the case.

The integral in this case is,

$$\begin{aligned}\int \sin(t)\left(4\cos^3(t)+6\cos^2(t)-8\right)dt &= -\int 4u^3+6u^2-8\,du\\ &= -\left(u^4+2u^3-8u\right)+c\\ &= -\left(\cos^4(t)+2\cos^3(t)-8\cos(t)\right)+c\end{aligned}$$

Again, be careful with the minus sign in front of the whole integral.

(c) In this integral, unlike the first one in this set, there are two terms and each will require a different substitution. So, to do this integral we'll first need to split up the integral as follows,

$$\int \mathbf{e}^{-z}+\sec^2\left(\frac{z}{10}\right)dz = \int \mathbf{e}^{-z}\,dz + \int \sec^2\left(\frac{z}{10}\right)dz$$

Here are the substitutions for each integral.

$$\begin{aligned} u&=-z & du&=-dz & &\Rightarrow & dz&=-du\\ v&=\frac{z}{10} & dv&=\frac{1}{10}dz & &\Rightarrow & dz&=10dv\end{aligned}$$

Notice that we used different letters for each substitution to avoid confusion when we go to plug back in for *u* and *v*.

Here is the integral.

$$\begin{aligned}\int \mathbf{e}^{-z}+\sec^2\left(\frac{z}{10}\right)dz &= -\int \mathbf{e}^{u}\,du+10\int\sec^2(v)\,dv\\ &= -\mathbf{e}^{u}+10\tan(v)+c\\ &= -\mathbf{e}^{-z}+10\tan\left(\frac{z}{10}\right)+c\end{aligned}$$

(d) As with the last problem this integral will require two separate substitutions. Let's first break up the integral.

$$\int \sin w\sqrt{1-2\cos w}+\frac{1}{7w+2}\,dw = \int \sin w\left(1-2\cos w\right)^{\frac{1}{2}}dw + \int \frac{1}{7w+2}\,dw$$

Here are the substitutions for this integral.

$$\begin{aligned} u&=1-2\cos(w) & du&=2\sin(w)\,dw & &\Rightarrow & \sin(w)\,dw&=\frac{1}{2}du\\ v&=7w+2 & dv&=7dw & &\Rightarrow & dw&=\frac{1}{7}dv\end{aligned}$$

The integral is then,

$$\begin{aligned}\int \sin w\sqrt{1-2\cos w}+\frac{1}{7w+2}\,dw &= \frac{1}{2}\int u^{\frac{1}{2}}\,du+\frac{1}{7}\int\frac{1}{v}\,dv\\ &= \frac{1}{2}\left(\frac{2}{3}\right)u^{\frac{3}{2}}+\frac{1}{7}\ln|v|+c\\ &= \frac{1}{3}\left(1-2\cos w\right)^{\frac{3}{2}}+\frac{1}{7}\ln|7w+2|+c\end{aligned}$$

(e) This integral is similar to the last two with one exception. This integral does need to be split into two integrals. However, this time this will be done because the first term doesn't need a substitution while the second term does.

$$\int x^2+\mathbf{e}^{1-x}\,dx=\int x^2\,dx+\int \mathbf{e}^{1-x}\,dx$$

Here is the substitution for the second integral.

$$u=1-x \qquad du=-dx \qquad \Rightarrow \qquad dx=-du$$

The integral is then,

$$\begin{aligned}\int x^2+\mathbf{e}^{1-x}\,dx &= \int x^2\,dx-\int \mathbf{e}^{u}\,du\\ &= \frac{1}{3}x^3-\mathbf{e}^{u}+c\\ &= \frac{1}{3}x^3-\mathbf{e}^{1-x}+c\end{aligned}$$

(f) The last problem in this set can be tricky. If there was just an x in the numerator we could do a quick substitution to get a natural logarithm. Likewise if there wasn't an x in the numerator we would get an inverse tangent after a quick substitution.

To get this integral into a form that we can work with we will first need to break it up as follows.

$$\begin{aligned}\int\frac{10x+3}{x^2+16}\,dx &= \int\frac{10x}{x^2+16}\,dx+\int\frac{3}{x^2+16}\,dx\\ &= \int\frac{10x}{x^2+16}\,dx+\frac{1}{16}\int\frac{3}{\frac{x^2}{16}+1}\,dx\end{aligned}$$

The substitutions for each of the integral above are,

$$u=x^2+16 \qquad du=2x\,dx \qquad \Rightarrow \qquad x\,dx=\frac{1}{2}du$$

$$v=\frac{x}{4} \qquad dv=\frac{1}{4}dx \qquad \Rightarrow \qquad dx=4dv$$

The integral is then,

$$\begin{aligned}\int \frac{10x+3}{x^2+16}\,dx &= 5\int \frac{1}{u}\,du + \frac{3}{4}\int \frac{1}{v^2+1}\,dv\\ &= 5\ln\left|u\right| + \frac{3}{4}\tan^{-1}\left(v\right) + c\\ &= 5\ln\left|x^2+16\right| + \frac{3}{4}\tan^{-1}\left(\frac{x}{4}\right) + c\end{aligned}$$

In this set of examples we saw several examples where the integral needed to be split into pieces that required different (or no) substitutions. Be on the look out for this. Missing this can make the problem very difficult.

The next set of examples of the substitution rule that we're going to take a look at is somewhat more difficult than those that we've done before.

Example 2 Evaluate each of the following integrals.

(a) $\int \tan x\,dx$

(b) $\int \sec y\,dy$

(c) $\displaystyle\int \frac{\cos\left(\sqrt{x}\right)}{\sqrt{x}}\,dx$

(d) $\int \mathbf{e}^{t+\mathbf{e}^t}\,dt$

(e) $\int 2x^3\sqrt{x^2+1}\,dx$

Solution

(a) The first question about this problem is probably why it's here. Substitution rule problems generally require more than a single function.

The key to this problem is to realize that there really are two functions here. All we need to do is remember the definition of tangent and we can write the integral as,

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$$

Written in this way we can see that the following substitution will work for us,

$$u = \cos x \qquad du = -\sin x\,dx \qquad \Rightarrow \qquad \sin x\,dx = -du$$

The integral is then,

$$\begin{aligned}\int \tan x\,dx &= -\int \frac{1}{u}\,dx\\ &= -\ln\left|u\right| + c\\ &= -\ln\left|\cos x\right| + c\end{aligned}$$

Now, while this is a perfectly serviceable answer that minus sign in front is liable to cause problems if we aren't careful. So, let's rewrite things a little. Recalling a property of logarithms we can move the minus sign in front to an exponent on the cosine and then do a little simplification.

$$\begin{aligned}\int \tan x\,dx &= -\ln\left|\cos x\right| + c\\ &= \ln\left|\cos x\right|^{-1} + c\\ &= \ln\frac{1}{\left|\cos x\right|} + c\\ &= \ln\left|\sec x\right| + c\end{aligned}$$

This is the formula that is typically given for the integral of tangent.

Note that we could integrate cotangent in a similar manner.

(b) This problem also at first appears to not belong in the substitution rule problems. This is even more of a problem upon noticing that we can't just use the definition of the secant function to write this in a form that will allow the use of the substitution rule.

This problem is going to require a technique that isn't used terribly often at this level, but is a useful technique to be aware of. We are going to write the integrand in the following way.

$$\int \sec y\,dy = \int \frac{\sec y}{1}\frac{\left(\sec y + \tan y\right)}{\left(\sec y + \tan y\right)}dy$$

First, we will think of the secant as a fraction and then multiply the top and bottom of the fraction by the same term. Doing this will actually allow us to use the substitution rule. To see how this will work let's simplify the integrand somewhat.

$$\int \sec y\,dy = \int \frac{\sec^2 y + \tan y \sec y}{\sec y + \tan y}dy$$

We can now use the following substitution.

$$u = \sec y + \tan y \qquad du = \left(\sec y \tan y + \sec^2 y\right)dy$$

The integral is then,

$$\begin{aligned}\int \sec y\,dy &= \int \frac{1}{u}dy\\ &= \ln\left|u\right| + c\\ &= \ln\left|\sec y + \tan y\right| + c\end{aligned}$$

Sometimes multiplying the top and bottom of a fraction by a carefully chosen term will allow us to work a problem.

We can use a similar process for integration cosecant.

(c) This next problem has a subtlety to it that can get us in trouble if we aren't paying attention. Because of the root in the cosine it makes some sense to use the following substitution.

$$u = x^{\frac{1}{2}} = \qquad du = \frac{1}{2}x^{-\frac{1}{2}}dx$$

This is where we need to be careful. Upon rewriting the differential we get,

$$2du = \frac{1}{\sqrt{x}}dx$$

The root that is in the denominator will not become a u as we might have been tempted to do. Instead it will get taken care of in the differential.

The integral is,

$$\begin{aligned}\int \frac{\cos\left(\sqrt{x}\right)}{\sqrt{x}}\,dx &= 2\int \cos\left(u\right)du \\ &= 2\sin\left(u\right)+c \\ &= 2\sin\left(\sqrt{x}\right)+c\end{aligned}$$

(d) With this problem we need to very carefully pick our substitution. As the problem is written we might be tempted to use the following substitution,

$$u = t+\mathbf{e}^{t} \qquad du = \left(1+\mathbf{e}^{t}\right)dt$$

However, this won't work. The differential doesn't show up anywhere in the integrand and we just wouldn't be able to eliminate the t's with this substitution.

In order to work this problem we will need to rewrite the integrand as follows,

$$\int \mathbf{e}^{t+\mathbf{e}^{t}}\,dt = \int \mathbf{e}^{t}\mathbf{e}^{\mathbf{e}^{t}}\,dt$$

We will now use the substitution,

$$u = \mathbf{e}^{t} \qquad du = \mathbf{e}^{t}dt$$

The integral is,

$$\begin{aligned}\int \mathbf{e}^{t+\mathbf{e}^{t}}\,dt &= \int \mathbf{e}^{u}\,dt \\ &= \mathbf{e}^{u}+c \\ &= \mathbf{e}^{\mathbf{e}^{t}}+c\end{aligned}$$

(e) This last problem in this set is different from all the other substitution problems that we've worked to this point. Given the fact that we've got more than an x under the root it makes sense that the substitution pretty much has to be,

$$u = x^2 + 1 \qquad\qquad du = 2x\,dx$$

However, if we use this substitution we will get the following,

$$\begin{aligned}\int 2x^3\sqrt{x^2+1}\,dx &= \int x^2\sqrt{x^2+1}\left(2x\right)dx \\ &= \int x^2 u^{\frac{1}{2}}\,du\end{aligned}$$

This is a real problem. Our integrals can't have two variables in them. Normally this would mean that we chose our substitution incorrectly. However, in this case we can write the substitution as follows,

$$x^2 = u - 1$$

and now, we can eliminate the remaining x from our integral. Doing this gives,

$$\begin{aligned}\int 2x^3\sqrt{x^2+1}\,dx &= \int \left(u-1\right)u^{\frac{1}{2}}\,du \\ &= \int u^{\frac{3}{2}} - u^{\frac{1}{2}}\,du \\ &= \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + c \\ &= \frac{2}{5}\left(x^2+1\right)^{\frac{5}{2}} - \frac{2}{3}\left(x^2+1\right)^{\frac{3}{2}} + c\end{aligned}$$

Sometimes, we will need to use a substitution more than once.

This final set of examples isn't too bad once you see the substitutions and that is the difficulty with this set.

Example 3 Evaluate each of the following integrals.

(a) $\displaystyle\int \frac{1}{x\ln x}\,dx$

(b) $\displaystyle\int \frac{\mathbf{e}^{2t}}{1+\mathbf{e}^{2t}}\,dt$

(c) $\displaystyle\int \frac{\mathbf{e}^{2t}}{1+\mathbf{e}^{4t}}\,dt$

(d) $\displaystyle\int \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx$

Solution

(a) In this case we know that we can't integrate a logarithm by itself and so it makes some sense (hopefully) that the logarithm will need to be in the substitution. Here is the substitution for this problem.

$$u = \ln x \qquad du = \frac{1}{x}dx$$

So the *x* in the denominator of the integrand will get substituted away with the differential. Here is the integral for this problem.

$$\begin{aligned}\int \frac{1}{x\ln x}dx &= \int \frac{1}{u}du \\ &= \ln|u| + c \\ &= \ln|\ln x| + c\end{aligned}$$

(b) Again, the substitution here is a little tricky. In this case the substitution is,

$$u = 1 + \mathbf{e}^{2t} \qquad du = 2\mathbf{e}^{2t}dt \qquad \Rightarrow \qquad \mathbf{e}^{2t}dt = \frac{1}{2}du$$

The integral is then,

$$\begin{aligned}\int \frac{\mathbf{e}^{2t}}{1+\mathbf{e}^{2t}}dt &= \frac{1}{2}\int \frac{1}{u}du \\ &= \frac{1}{2}\ln\left|1+\mathbf{e}^{2t}\right| + c\end{aligned}$$

(c) In this case we can't use the same type of substitution that we used in the previous problem. In order to use the substitution in the previous example the exponential in the numerator and the denominator need to be the same and in this case they aren't.

To see the correct substitution for this problem note that,

$$\mathbf{e}^{4t} = \left(\mathbf{e}^{2t}\right)^2$$

Using this, the integral can be written as follows,

$$\int \frac{\mathbf{e}^{2t}}{1+\mathbf{e}^{4t}}dt = \int \frac{\mathbf{e}^{2t}}{1+\left(\mathbf{e}^{2t}\right)^2}dt$$

We can now use the following substitution.

$$u = \mathbf{e}^{2t} \qquad du = 2\mathbf{e}^{2t}dt \qquad \Rightarrow \qquad \mathbf{e}^{2t}dt = \frac{1}{2}du$$

The integral is then,

$$\int \frac{\mathbf{e}^{2t}}{1+\mathbf{e}^{4t}}\,dt = \frac{1}{2}\int \frac{1}{1+u^2}\,du$$
$$= \frac{1}{2}\tan^{-1}\left(u\right)+c$$
$$= \frac{1}{2}\tan^{-1}\left(\mathbf{e}^{2t}\right)+c$$

(d) This integral is similar to the first problem in this set. Since we don't know how to integrate inverse sine functions it seems likely that this will be our substitution. If we use this as our substitution we get,

$$u = \sin^{-1}\left(x\right) \qquad\qquad du = \frac{1}{\sqrt{1-x^2}}\,dx$$

So, the root in the integral will get taken care of in the substitution process and this will eliminate all the *x*'s from the integral. Therefore this was the correct substitution.

The integral is,

$$\int \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx = \int u\,du$$
$$= \frac{1}{2}u^2 + c$$
$$= \frac{1}{2}\left(\sin^{-1}x\right)^2 + c$$

Over the last couple of sections we've seen a lot of substitution rule examples. There are a couple of general rules that we will need to remember when doing these problems. First, when doing a substitution remember that when the substitution is done all the *x*'s in the integral (or whatever variable is being used for that particular integral) should all be substituted away. This includes the *x* in the *dx*. After the substitution only *u*'s should be left in the integral. Also, sometimes the correct substitution is a little tricky to find and more often than not there will need to be some manipulation of the differential or integrand in order to actually do the substitution.

Many integrals will require us to break them up so we can do multiple substitutions so be on the lookout for those kinds of integrals/substitutions.

Area Problem

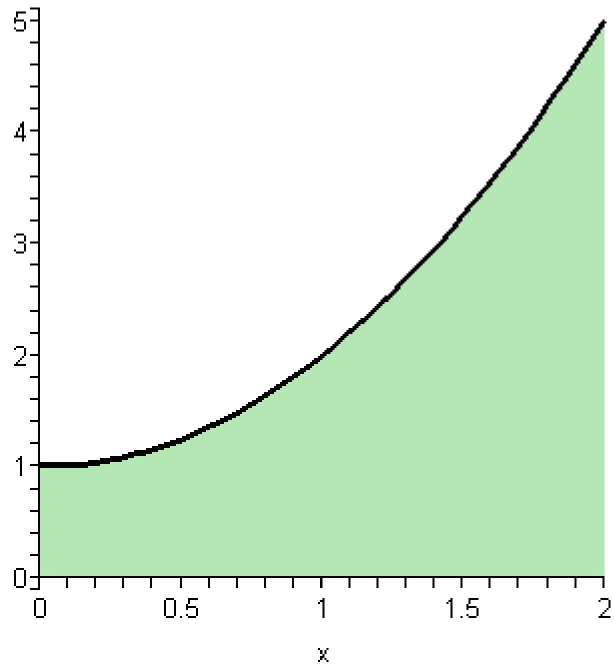
As noted in the first section there are two kinds of integrals. To this point in this chapter we've looked at indefinite integrals. It is now time to start thinking about the second kind of integral : Definite Integrals.

However, before we do that we're going to take a look at the Area Problem. The area problem is to definite integrals what the tangent and rate of change problems are to derivatives.

The area problem will give us one of the interpretations of a definite integral and it will lead us to the definition of the definite integral.

To start off we are going to assume that we've got a function *f(x)* that is positive on some interval [*a*,*b*]. What we want to do is determine the area of the region between the function and the *x*-axis.

It's probably easiest to see how we do this with an example. So let's determine the area between $f(x) = x^2 + 1$ on [0,2]. In other words, we want to determine the area of the shaded region below.

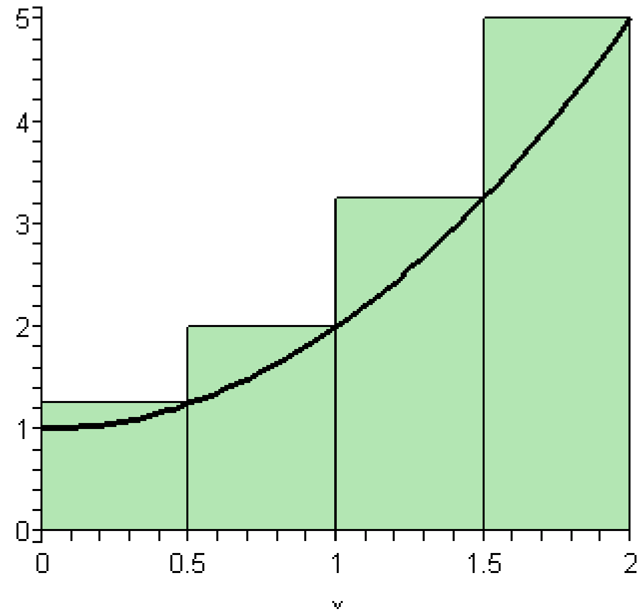


Now, at this point, we can't do this exactly. However, we can estimate the area. We will estimate the area by dividing up the interval into *n* subintervals each of width,

$$\Delta x = \frac{b-a}{n}$$

Then in each interval we can form a rectangle whose height is given by the function value at a specific point in the interval. We can then find the area of each of these rectangles, add them up and this will be an estimate of the area.

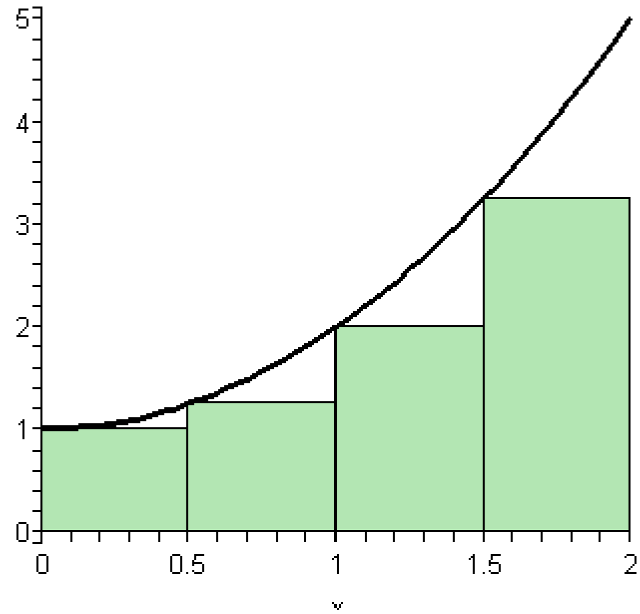
It's probably easier to see this with a sketch of the situation. So, let's divide up the interval into 4 subintervals and use the function value at the right endpoint of each interval to define the height of the rectangle. This gives,



Note that by choosing the height as we did each of the rectangles will over estimate the area since each rectangle takes in more area than the graph each time. Let's find the estimated area. First, the width of each of the rectangles is $\frac{1}{2}$. The height of each rectangle is determined by the function value at the right endpoint and so the height of each rectangle is nothing more that the function value at the right endpoint. Here is the estimated area.

$$\begin{aligned} A_r &= \frac{1}{2} f\left(\frac{1}{2}\right) + \frac{1}{2} f(1) + \frac{1}{2} f\left(\frac{3}{2}\right) + \frac{1}{2} f(2) \\ &= \frac{1}{2}\left(\frac{5}{4}\right) + \frac{1}{2}(2) + \frac{1}{2}\left(\frac{13}{4}\right) + \frac{1}{2}(5) \\ &= 5.75 \end{aligned}$$

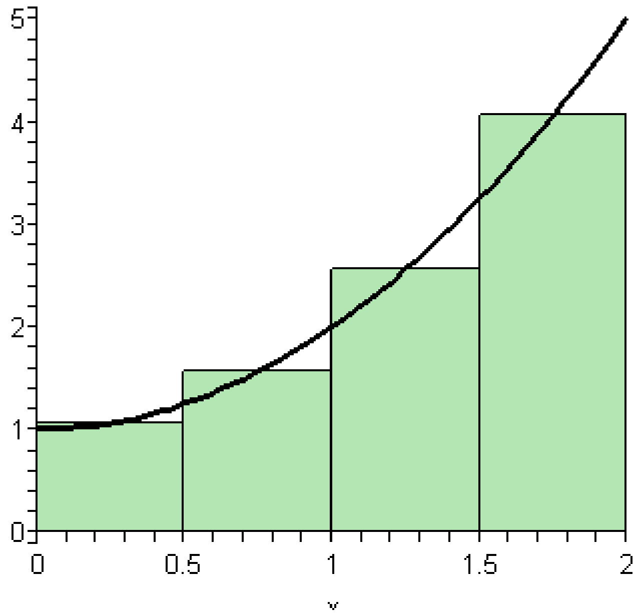
Of course taking the rectangle heights to be the function value at the right endpoint is not our only option. We could have taken the rectangle heights to be the function value at the left endpoint. Using the left endpoints as the heights of the rectangles will give the following graph and estimated area.



$$
\begin{aligned}
A_l &= \frac{1}{2} f(0) + \frac{1}{2} f\left(\frac{1}{2}\right) + \frac{1}{2} f(1) + \frac{1}{2} f\left(\frac{3}{2}\right) \\
&= \frac{1}{2}(1) + \frac{1}{2}\left(\frac{5}{4}\right) + \frac{1}{2}(2) + \frac{1}{2}\left(\frac{13}{4}\right) \\
&= 3.75
\end{aligned}
$$

In this case we can see that the estimation will be an underestimation since each rectangle misses some of the area each time.

There is one more common point for getting the heights of the rectangles that is often more accurate. Instead of using the right or left endpoints of each sub interval we could take the midpoint of each subinterval as the height of each rectangle. Here is the graph for this case.



So, it looks like each rectangle will over and under estimate the area. This means that the approximation this time should be much better. Here is the estimation for this case.

$$\begin{aligned} A_m &= \frac{1}{2} f\left(\frac{1}{4}\right) + \frac{1}{2} f\left(\frac{3}{4}\right) + \frac{1}{2} f\left(\frac{5}{4}\right) + \frac{1}{2} f\left(\frac{7}{4}\right) \\ &= \frac{1}{2}\left(\frac{17}{16}\right) + \frac{1}{2}\left(\frac{25}{16}\right) + \frac{1}{2}\left(\frac{41}{16}\right) + \frac{1}{2}\left(\frac{65}{16}\right) \\ &= 4.625 \end{aligned}$$

We've now got three estimates. For comparison's sake the exact area is

$$A = \frac{14}{3} = 4.66\overline{6}$$

So, both the right and left endpoint estimation did not do all that great of a job at the estimation. The midpoint estimation however did quite well.

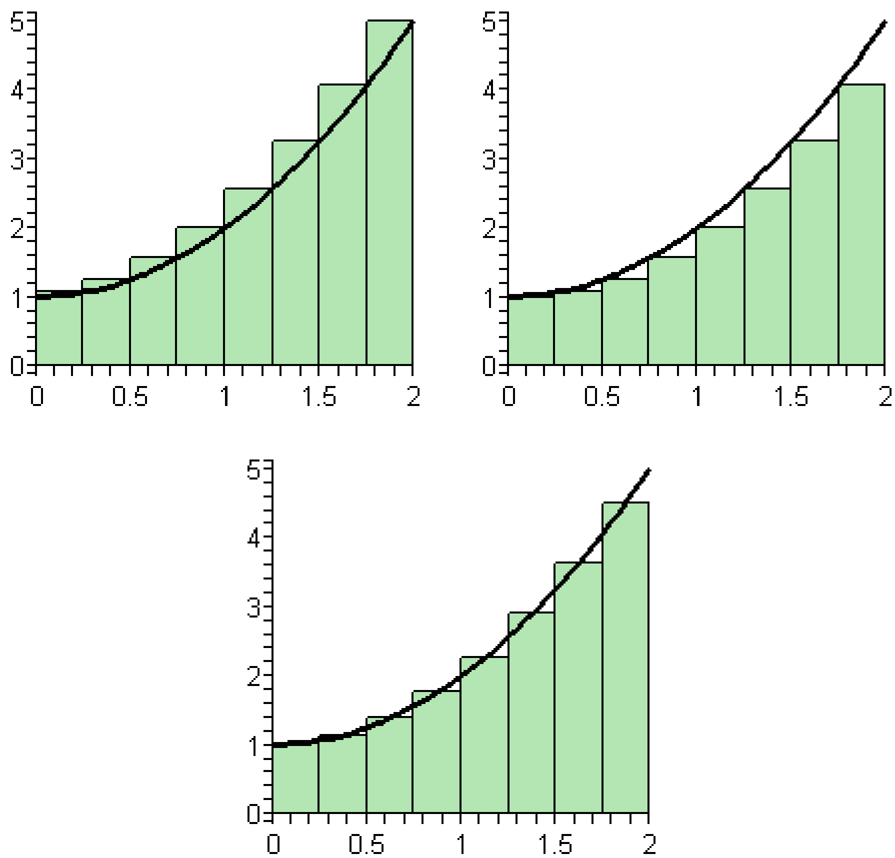
Be careful to not draw any conclusion about how choosing each of the points will affect our estimation. In this case, because we are working with an increasing function choosing the right endpoints will overestimate and choosing left endpoint will underestimate.

If we were to work with a decreasing function we would get the opposite results. In the case of a decreasing function the right endpoints will underestimate and the left endpoints will overestimate.

Also, if we had a function that both increased and decreased in the interval we would, in all likelihood, not even be able to determine if we would get an overestimation or underestimation.

Now, let's suppose that we want a better estimation, because none of the estimations above really did all that great of a job at estimating the area. We could try to find a different point to use for the height of each rectangle but that would be cumbersome and there wouldn't be any guarantee that the estimation would in fact be better. Also, we would like a method for getting better approximations that would work for any function we would chose to work with and if we just pick new points that may not work for other functions.

The easiest way to get a better approximation is to take more rectangles (*i.e.* increase n). Let's double the number of rectangles that we used and see what happens. Here are the graphs showing the eight rectangles and the estimations for each of the three choices for rectangle heights that we used above.



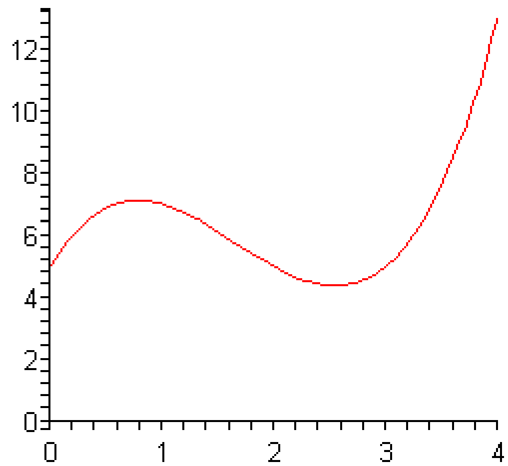
$A_r = 5.1875$ $\qquad$ $A_l = 4.1875$ $\qquad$ $A_m = 4.65625$

So, increasing the number of rectangles did improve the accuracy of the estimation.

Let's work a slightly more complicated example.

Example 1 Estimate the area between $f(x) = x^3 - 5x^2 + 6x + 5$ and the *x*-axis using *n*=5 subintervals and all three cases above for the function heights.

Solution
First, let's get the graph to make sure that the function is positive.



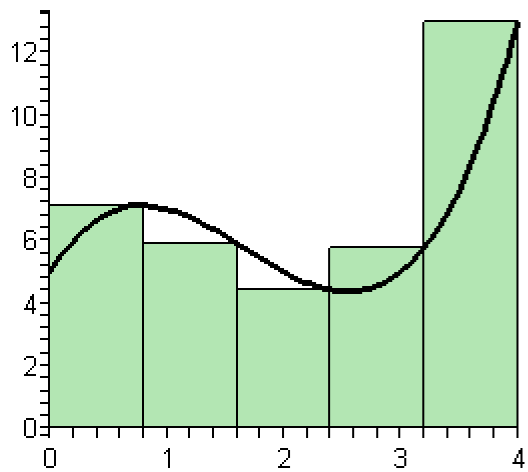
So, the graph is positive and the width of each subinterval will be,

$$\Delta x = \frac{4}{5} = 0.8$$

This means that the endpoints of the subintervals are,

$$0, \; 0.8, \; 1.6, \; 2.4, \; 3.2, \; 4$$

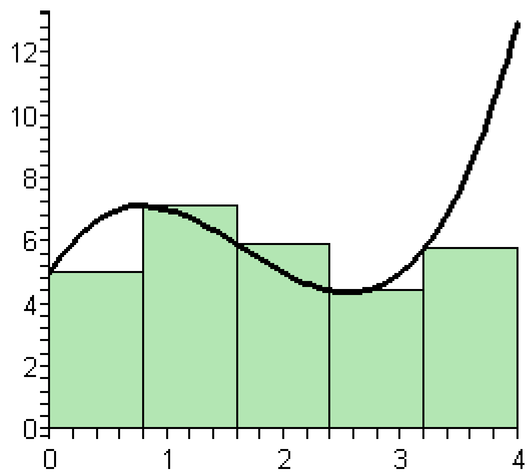
Let's first look at using the right endpoints for the function height. Here is the graph for this case.



Notice, that unlike the first area we looked at, the choosing the right endpoints here will both over and underestimate the area. The area estimation is,

$$\begin{aligned} A_r &= 0.8f\left(0.8\right)+0.8f\left(1.6\right)+0.8f\left(2.4\right)+0.8f\left(3.2\right)+0.8f\left(4\right) \\ &= 28.96 \end{aligned}$$

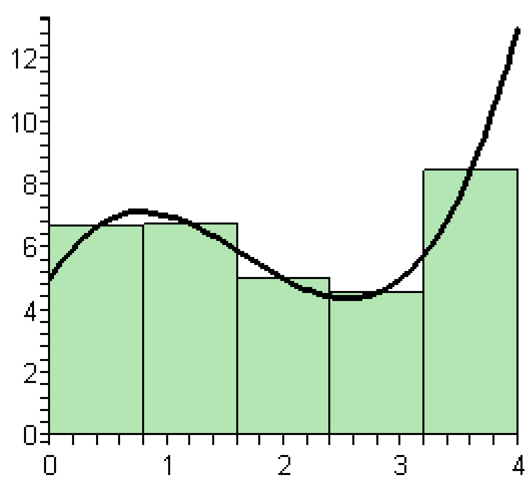
Now let's take a look at left endpoints for the function height. Here is the graph.



The area estimation is,

$$\begin{aligned} A_r &= 0.8f\left(0\right)+0.8f\left(0.8\right)+0.8f\left(1.6\right)+0.8f\left(2.4\right)+0.8f\left(3.2\right) \\ &= 22.56 \end{aligned}$$

Finally, let's take a look at the midpoints for the heights of each rectangle.



The area estimation is,

$$\begin{aligned} A_r &= 0.8f\left(0.4\right)+0.8f\left(1.2\right)+0.8f\left(2\right)+0.8f\left(2.8\right)+0.8f\left(3.6\right) \\ &= 25.12 \end{aligned}$$

For comparison purposes the exact area is,

$$A = \frac{76}{3} = 25.33\overline{3}$$

So, again the midpoint did a better job than the other two. While this will be the case more often than not, it won't always be the case and so don't expect this to always happen.

Now, let's move on to the general case. Let's start out with $f(x)>0$ on $[a,b]$ and we'll divide the interval into n subintervals each of length,

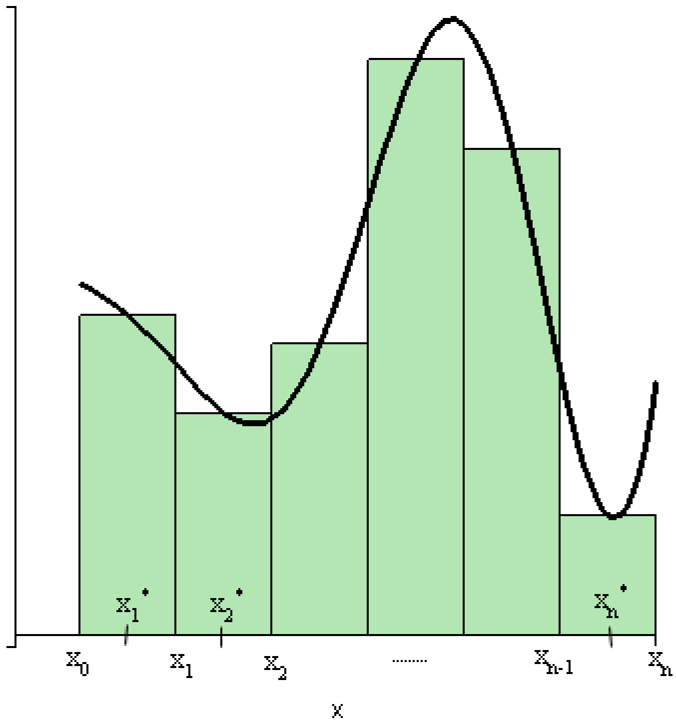
$$\Delta x = \frac{b-a}{n}$$

Note that the subintervals don't have to be equal length, but it will make our work significantly easier. The endpoints of each subinterval are,

$$\begin{aligned} x_0 &= a \\ x_1 &= a + \Delta x \\ x_2 &= a + 2\Delta x \\ &\vdots \\ x_i &= a + i\Delta x \\ &\vdots \\ x_{n-1} &= a + (n-1)\Delta x \\ x_n &= a + n\Delta x = b \end{aligned}$$

Next in each interval $[x_0, x_1], [x_1, x_2], \ldots, [x_{i-1}, x_i], \ldots, [x_{n-1}, x_n]$ we choose a point $x_0^*, x_1^*, \ldots, x_i^*, \ldots x_n^*$. These points will define the height of the rectangle in each subinterval. Note as well that these points do not have to occur at the same point in each subinterval.

Here is a sketch of this situation.



The area under the curve on the given interval is then approximately,

$$A \approx f\left(x_0^*\right)\Delta x + f\left(x_1^*\right)\Delta x + \cdots + f\left(x_i^*\right)\Delta x + \cdots + f\left(x_n^*\right)\Delta x$$

We will use **summation notation** or **sigma notation** at this point to simplify up our notation a little. If you need a refresher on summation notation check out the section devoted to this in the Extras chapter.

Using summation notation the area estimation is,

$$A \approx \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$$

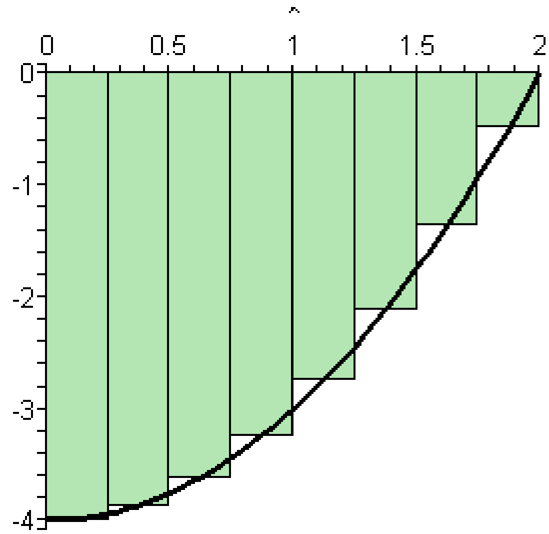
The summation in the above equation is called a **Riemann Sum**.

To get a better estimation we will take *n* larger and larger. In fact, if we let *n* go out to infinity we will get the exact area. In other words,

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$$

Before leaving this section let's address one more issue. To this point we've required the function to be positive in our work. Many functions are not strictly positive however.

Consider the case of $f(x) = x^2 - 4$ on [0,2]. If we use *n=8* and the midpoints for the rectangle height we get the following graph.



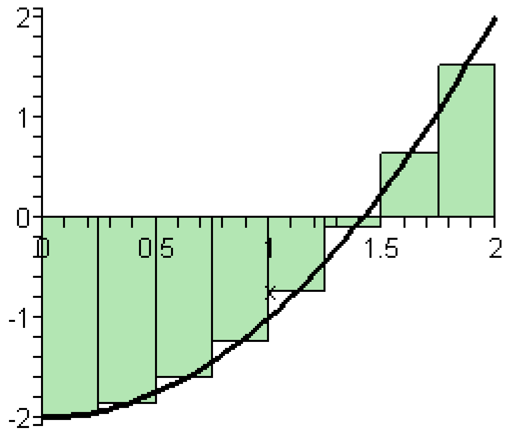
In this case let's notice that the function lies completely below the *x*-axis. The area estimation is,

$$\begin{aligned} A_m &= \frac{1}{4} f\left(\frac{1}{8}\right) + \frac{1}{4} f\left(\frac{3}{8}\right) + \frac{1}{4} f\left(\frac{5}{8}\right) \frac{1}{4} f\left(\frac{7}{8}\right) + \frac{1}{4} f\left(\frac{9}{8}\right) + \\ &\qquad \frac{1}{4} f\left(\frac{11}{8}\right) + \frac{1}{4} f\left(\frac{13}{8}\right) + \frac{1}{4} f\left(\frac{15}{8}\right) \\ &= -5.34375 \end{aligned}$$

Our answer is negative. This shouldn't be too surprising since all the function evaluations are negative.

So, using the technique in this section it looks like if the function is above the *x*-axis we will get a positive area and if the function is below the *x*-axis we will get a negative area.

Now, what about a function that is both positive and negative in the interval? For example, $f(x) = x^2 - 2$ on [0,2]. Using *n=8* and the midpoints of each interval the graph is,



Some of the rectangles are below the x-axis and so will give negative areas while some are above the x-axis and will give positive areas. Since more rectangles are below the x-axis than above it looks like we should probably get a negative area estimation for this case. In fact that is correct. Here the area estimation for this case.

$$\begin{aligned} A_m &= \frac{1}{4} f\left(\frac{1}{8}\right) + \frac{1}{4} f\left(\frac{3}{8}\right) + \frac{1}{4} f\left(\frac{5}{8}\right)\frac{1}{4} f\left(\frac{7}{8}\right) + \frac{1}{4} f\left(\frac{9}{8}\right) + \\ &\qquad \frac{1}{4} f\left(\frac{11}{8}\right) + \frac{1}{4} f\left(\frac{13}{8}\right) + \frac{1}{4} f\left(\frac{15}{8}\right) \\ &= -1.34375 \end{aligned}$$

In cases where the function is both above and below the x-axis the technique given in the section will give the *net* area between the function and the x-axis with areas below the x-axis negative and areas above the x-axis positive.

The Definition of the Definite Integral

In this section we will formally define the definite integral and give many of the properties of definite integrals.

Let's start off with the definition of a definite integral.

Definite Integral

Given a function *f(x)* that is continuous on the interval [*a*,*b*] we divide the interval into *n* subintervals of equal width, Δx, and from each interval choose a point, x_i^*. Then the **definite integral of *f(x)* from *a* to *b*** is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$$

The definite integral is defined to be exactly the limit and summation that we looked at in the last section to find the net area between a function and the x-axis. Also note that the notation for the definite integral is very similar to the notation for an indefinite integral. The reason for this will be apparent eventually.

There is also a little bit of terminology that we should get out of the way here. The number "a" that is at the bottom of the integral sign is called the **lower limit** of the integral and the number "b" at the top of the integral sign is called the **upper limit** of the integral. Also, despite the fact that a and b were given as an interval the lower limit will not necessarily be smaller than the upper limit.

Let's work a quick example. This example will use many of the properties and facts from the brief review of summation notation in the Extras chapter.

Example 1 Using the definition compute the following definite integral.

$$\int_0^2 x^2 + 1\,dx$$

Solution

First, we can't actually use the definition unless we determine which points in each interval that well use for x_i^*. In order to make our life easier we'll use the right endpoints of each interval.

From the previous section we know that for a general n the width of each subinterval is,

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}$$

The subintervals are then,

$$\left[0,\frac{2}{n}\right],\ \left[\frac{2}{n},\frac{4}{n}\right],\ \left[\frac{4}{n},\frac{6}{n}\right], \ldots, \left[\frac{2(i-1)}{n},\frac{2i}{n}\right], \ldots, \left[\frac{2(n-1)}{n},2\right]$$

As we can see the right endpoint of the i^{th} subinterval is

$$x_i^* = \frac{2i}{n}$$

The summation in the definition of the definite integral is then,

$$\begin{aligned} \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x &= \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right) \\ &= \sum_{i=1}^{n}\left(\left(\frac{2i}{n}\right)^2 + 1\right)\left(\frac{2}{n}\right) \\ &= \sum_{i=1}^{n}\left(\frac{8i^2}{n^3} + \frac{2}{n}\right) \end{aligned}$$

Now, we are going to have to take a limit of this. That means that we are going to need to "evaluate" this summation. In other words, we are going to have to use the formulas given in the summation notation review to eliminate the actual summation and get a formula for this for a general n.

To do this we will need to recognize that n is a constant as far as the summation notation is concerned. As we cycle through the integers from 1 to n in the summation only i changes and so anything that isn't an i will be a constant and can be factored out of the summation. In particular any n that is in the summation can be factored out if we need to.

Here is the summation "evaluation".

$$\begin{aligned}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x &= \sum_{i=1}^{n} \frac{8i^2}{n^3} + \sum_{i=1}^{n} \frac{2}{n} \\ &= \frac{8}{n^3}\sum_{i=1}^{n} i^2 + \frac{1}{n}\sum_{i=1}^{n} 2 \\ &= \frac{8}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{1}{n}(2n) \\ &= \frac{4(n+1)(2n+1)}{3n^2} + 2 \\ &= \frac{14n^2+12n+4}{3n^2}\end{aligned}$$

We can now compute the definite integral.

$$\begin{aligned}\int_0^2 x^2+1\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x \\ &= \lim_{n\to\infty}\frac{14n^2+12n+4}{3n^2} \\ &= \frac{14}{3}\end{aligned}$$

We've seen several methods for dealing with the limit in this problem so I'll leave it to you to verify the results.

Wow, that was a lot of work for a fairly simple function. There is a much simpler way of evaluating these and we will get to it eventually. The main purpose to this section is to get the main properties and facts about the definite integral out of the way.

So, let's start taking a look at some of the properties of the definite integral.

Properties

1. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$. We can interchange the limits on any definite integral, all that we need to do is tack a minus sign onto the integral when we do.

2. $\int_a^a f(x)\,dx = 0$. If the upper and lower limits are the same then there is no work to do, the integral is zero.
3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where c is any number. So, as with limits, derivatives, and indefinite integrals we can factor out a constant.
4. $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$. We can break up definite integrals across a sum or difference.
5. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ where c is any number. This property is more important than we might realize at first. One of the main uses of this property is to tell us how we can integrate a function over the adjacent intervals, $[a,c]$ and $[c,b]$. Note however that c doesn't need to be between a and b.
6. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$. The point of this property is to notice that as long as the function and limits are the same the variable of integration that we use in the definite integral won't affect the answer.

Let's do a couple of examples dealing with these properties.

Example 2 Use the results from the first example to evaluate each of the following.

(a) $\int_2^0 x^2 + 1\,dx$

(b) $\int_0^2 10x^2 + 10\,dx$

(c) $\int_0^2 t^2 + 1\,dt$

Solution

All of these will rely on the fact that,

$$\int_0^2 x^2 + 1\,dx = \frac{14}{3}$$

(a) In this case the only difference between the two is that the limits have interchanged. So, using the first property gives,

$$\int_2^0 x^2 + 1\,dx = -\int_0^2 x^2 + 1\,dx$$

$$= -\frac{14}{3}$$

(b) For this part notice that we can factor a 10 out of both terms and then out of the integral using the third property.

$$\int_0^2 10x^2 + 10\,dx = \int_0^2 10\left(x^2+1\right)dx$$
$$= 10\int_0^2 x^2 + 1\,dx$$
$$= 10\left(\frac{14}{3}\right)$$
$$= \frac{140}{3}$$

(c) In this case the only difference is the letter used and so this is just going to use property 6.

$$\int_0^2 t^2 + 1\,dt = \int_0^2 x^2 + 1\,dx = \frac{14}{3}$$

Here are a couple of examples using the other properties.

Example 3 Evaluate the following definite integral.

$$\int_{130}^{130} \frac{x^3 - x\sin\left(x\right) + \cos\left(x\right)}{x^2+1}\,dx$$

Solution

There really isn't anything to do with this integral once we notice that the limits are the same. Using the second property this is,

$$\int_{130}^{130} \frac{x^3 - x\sin\left(x\right) + \cos\left(x\right)}{x^2+1}\,dx = 0$$

Example 4 Given that $\int_6^{-10} f\left(x\right)dx = 23$ and $\int_{-10}^{6} g\left(x\right)dx = -9$ determine the value of

$$\int_{-10}^{6} 2f\left(x\right) - 10g\left(x\right)dx$$

Solution

We will first need to use the fourth property to break up the integral and the third property to factor out the constants.

$$\int_{-10}^{6} 2f\left(x\right) - 10g\left(x\right)dx = \int_{-10}^{6} 2f\left(x\right)dx - \int_{-10}^{6} 10g\left(x\right)dx$$
$$= 2\int_{-10}^{6} f\left(x\right)dx - 10\int_{-10}^{6} g\left(x\right)dx$$

Now notice that the limits on the first integral are interchanged with the limits on the given integral so switch them using the first property above. Once this is done we can plug in the known values of the integrals.

$$\int_{-10}^{6} 2f\left(x\right) - 10g\left(x\right)dx = -2\int_{6}^{-10} f\left(x\right)dx - 10\int_{-10}^{6} g\left(x\right)dx$$
$$= -2\left(23\right) - 10\left(-9\right)$$
$$= 44$$

Example 5 Given that $\int_{12}^{-10} f(x)\,dx = 6$, $\int_{100}^{-10} f(x)\,dx = -2$, and $\int_{100}^{-5} f(x)\,dx = 4$ determine the value of $\int_{-5}^{12} f(x)\,dx$.

Solution
This example is mostly an example of property 5 although there are a couple of uses of property 1 in the solution as well.

We need to figure out how to correctly break up the integral using property 5 to allow us to use the given pieces of information. First we'll note that there is an integral that has a "-5" in one of the limits. It's not the lower limit, but we can use property 1 to correct that eventually. The other limit is 100 so this is the number *c* that we'll use in property 5.

$$\int_{-5}^{12} f(x)\,dx = \int_{-5}^{100} f(x)\,dx + \int_{100}^{12} f(x)\,dx$$

We'll be able to get the value of the first integral, but the second still isn't in the list of know integrals. However, we do have second limit that has a limit of 100 in it. The other limit for this second integral is -10 and this will be *c* in this application of property 5.

$$\int_{-5}^{12} f(x)\,dx = \int_{-5}^{100} f(x)\,dx + \int_{100}^{-10} f(x)\,dx + \int_{-10}^{12} f(x)\,dx$$

At this point all that we need to do is use the property 1 on the first and third integral to get the limits to match up with the known integrals. After that we can plug in for the known integrals.

$$\begin{aligned}\int_{-5}^{12} f(x)\,dx &= -\int_{100}^{-5} f(x)\,dx + \int_{100}^{-10} f(x)\,dx - \int_{12}^{-10} f(x)\,dx \\ &= -4 - 2 - 6 \\ &= -12\end{aligned}$$

There are also some nice properties that we can use in comparing the general size of definite integrals. Here they are.

More Properties

7. If $f(x) \ge g(x)$ for $a \le x \le b$ then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$
8. If $f(x) \ge 0$ for $a \le x \le b$ then $\int_a^b f(x)\,dx \ge 0$. If you think about it a little you can see that this is really a special case of the previous property.
9. If $m \le f(x) \le M$ for $a \le x \le b$ then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$
10. $\left|\int_a^b f(x)\,dx\right| \le \int_a^b \left|f(x)\right|\,dx$

Interpretation of Definite Integral

As we alluded to in the previous section one possible interpretation of the definite integral is to give the net area between *f(x)* and the *x*-axis on the interval [*a*,*b*]. So, the net area between $f(x) = x^2 + 1$ and the *x*-axis on [0,2] is,

$$\int_0^2 x^2 + 1\,dx = \frac{14}{3}$$

If you look back in the last section this was the exact area that was given for the initial set of problems that we looked at in this area.

Net area is not the only possible interpretation, but it is one of the more common interpretations and it is an easy interpretation to visualize. We will see another interpretation in the next chapter.

Fundamental Theorem of Calculus, Part I

As noted this is only the first part to the Fundamental Theorem of Calculus. We will give the second part in the next section as it is the key to easily computing definite integrals and that is the subject of the next section.

The first part of the Fundamental Theorem of Calculus tells us how to differentiate certain types of definite integrals.

Fundamental Theorem of Calculus, Part I

If *f(x)* is continuous on [*a*,*b*] then,

$$g(x) = \int_a^x f(t)\,dt$$

is also continuous on [*a*,*b*] and it can be differentiated. The derivative is,

$$g'(x) = f(x)$$

An alternate notation for the derivative portion of this is,

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Example 6 Differentiate each of the following.

(a) $g(x) = \int_{-4}^{x} \mathbf{e}^{2t} \cos^2(1-5t)\,dt$

(b) $\int_{x^2}^{1} \frac{t^4+1}{t^2+1}\,dt$

Solution

(a) This one is nothing more than a quick application of the Fundamental Theorem of Calculus, Part I.

$$g'(x) = \mathbf{e}^{2x}\cos^2(1-5x)$$

(b) This one needs some work before we can use the Fundamental Theorem of Calculus.

The first thing to notice is that the FToC requires the lower limit to be a constant and the upper limit to be the variable. So, using a property of definite integrals we can rewrite the integral.

$$\frac{d}{dx}\int_{x^2}^{1}\frac{t^4+1}{t^2+1}\,dt = \frac{d}{dx}\left(-\int_{1}^{x^2}\frac{t^4+1}{t^2+1}\,dt\right) = -\frac{d}{dx}\int_{1}^{x^2}\frac{t^4+1}{t^2+1}\,dt$$

The next thing to notice is that the FToC also requires an x in the upper limit of integration and we've got x^2. To do this derivative we're going to need the following version of the chain rule.

$$\frac{d}{dx}\left(g\left(u\right)\right) = \frac{d}{du}\left(g\left(u\right)\right)\frac{du}{dx} \qquad\qquad \text{where } u = f\left(x\right)$$

So, if $u = x^2$ we get the following for the derivative.

$$\begin{aligned}\frac{d}{dx}\int_{x^2}^{1}\frac{t^4+1}{t^2+1}\,dt &= -\frac{d}{dx}\int_{1}^{x^2}\frac{t^4+1}{t^2+1}\,dt \\ &= -\frac{d}{du}\int_{1}^{u}\frac{t^4+1}{t^2+1}\,dt\,\frac{du}{dx} \qquad\qquad \text{where } u = x^2 \\ &= -\frac{u^4+1}{u^2+1}\left(2x\right) \\ &= -2x\frac{u^4+1}{u^2+1}\end{aligned}$$

The final step is to get everything back in terms of x.

$$\begin{aligned}\frac{d}{dx}\int_{x^2}^{1}\frac{t^4+1}{t^2+1}\,dt &= -2x\frac{\left(x^2\right)^4+1}{\left(x^2\right)^2+1} \\ &= -2x\frac{x^8+1}{x^4+1}\end{aligned}$$

Using the chain rule as we did in the last part of this example we can derive some general formulas for some more complicated problems.

First,

$$\frac{d}{dx}\int_{a}^{u(x)} f\left(t\right)dt = u'\left(x\right)f\left(u\left(x\right)\right)$$

This is simply the chain rule for these kinds of problems.

Next, we can get a formula for integrals in which the upper limit is a constant and the lower limit is a function of x.

$$\frac{d}{dx}\int_{v(x)}^{b} f(t)\,dt = -\frac{d}{dx}\int_{b}^{v(x)} f(t)\,dt = -v'(x)\,f(v(x))$$

Finally, we can also get a version for both limits being functions of *x*.

$$\begin{aligned}\frac{d}{dx}\int_{v(x)}^{u(x)} f(t)\,dt &= \frac{d}{dx}\left(\int_{v(x)}^{0} f(t)\,dt + \int_{0}^{u(x)} f(t)\,dt\right)\\ &= -v'(x)\,f(v(x)) + u'(x)\,f(u(x))\end{aligned}$$

Example 7 Differentiate the following integral.

$$\int_{\sqrt{x}}^{3x} t^2 \sin\left(1+t^2\right) dt$$

Solution

This will use the final formula that we derived above.

$$\begin{aligned}\frac{d}{dx}\int_{\sqrt{x}}^{3x} t^2 \sin\left(1+t^2\right) dt &= -\frac{1}{2}x^{-\frac{1}{2}}\left(\sqrt{x}\right)^2 \sin\left(1+\left(\sqrt{x}\right)^2\right) + (3)(3x)^2 \sin\left(1+(3x)^2\right)\\ &= -\frac{1}{2}\sqrt{x}\sin(1+x) + 27x^2\sin\left(1+9x^2\right)\end{aligned}$$

Computing Definite Integrals

In this section we are going to concentrate on how we actually evaluate definite integrals. To do this we will need the Fundamental Theorem of Calculus, Part II.

Fundamental Theorem of Calculus, Part II

Suppose *f(x)* is a continuous function on [*a*,*b*] and also suppose that *F(x)* is any anti-derivative for *f(x)*. Then,

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

Recall that when we talk about an anti-derivative for a function we are really talking about the indefinite integral for the function. So, to evaluate a definite integral the first thing that we're going to do is evaluate the indefinite integral for the function. This should explain the similarity in the notations for the indefinite and definite integrals.

Also notice that we require the function to be continuous. This was also a requirement in the definition of the definite integral. We didn't make a big deal about this in the last section. In this section however, we will need to keep this condition in mind as we do our evaluations.

Next let's address the fact that we can use any anti-derivative of *f(x)* in the evaluation. Let's take a final look at the following integral.

$$\int_0^2 x^2 + 1\,dx$$

Both of the following are anti-derivatives of the integrand.

$$F(x) = \frac{1}{3}x^3 + x \qquad \text{and} \qquad F(x) = \frac{1}{3}x^3 + x - \frac{18}{31}$$

Using the FToC to evaluate this integral using the first anti-derivatives gives,

$$\begin{aligned}\int_0^2 x^2 + 1\,dx &= \left(\frac{1}{3}x^3 + x\right)\Bigg|_0^2 \\ &= \frac{1}{3}(2)^3 + 2 - \left(\frac{1}{3}(0)^3 + 0\right) \\ &= \frac{14}{3}\end{aligned}$$

Much easier than using the definition wasn't it? Let's now use the second anti-derivative to evaluate this definite integral.

$$\begin{aligned}\int_0^2 x^2 + 1\,dx &= \left(\frac{1}{3}x^3 + x - \frac{18}{31}\right)\Bigg|_0^2 \\ &= \frac{1}{3}(2)^3 + 2 - \frac{18}{31} - \left(\frac{1}{3}(0)^3 + 0 - \frac{18}{31}\right) \\ &= \frac{14}{3} - \frac{18}{31} + \frac{18}{31} \\ &= \frac{14}{3}\end{aligned}$$

The constant that we tacked onto the second anti-derivative canceled in the evaluation step. So, when choosing the anti-derivative to use in the evaluation process make your life easier and don't bother with the constant as it will only end up canceling in the long run.

Also, note that we're going to have to be very careful with minus signs and parenthesis with these problems. It's very easy to get in a hurry and mess them up.

Let's work some examples.

Example 1 Evaluate each of the following.

(a) $\int y^2 + y^{-2}\,dy$

(b) $\int_1^2 y^2 + y^{-2}\,dy$

(c) $\int_{-1}^2 y^2 + y^{-2}\,dy$

Solution

(a) This is the only indefinite integral in this section and by now we should be getting pretty good with these so we won't spend a lot of time on this part.

$$\int y^2 + y^{-2}\,dy = \frac{1}{3}y^3 - y^{-1} + c$$

(b) Recall from our first example above that all we really need here is any anti-derivative of the integrand. This is the reason for the first problem in this set. Again, recall that the indefinite integral is the most general anti-derivative of the integrand. Also as noted above any constants we tack on will just cancel in the long run and so we'll use the answer from (a) without the "+*c*".

Here's the integral,

$$\begin{aligned}\int_1^2 y^2 + y^{-2}\,dy &= \left.\left(\frac{1}{3}y^3 - \frac{1}{y}\right)\right|_1^2 \\ &= \frac{1}{3}(2)^3 - \frac{1}{2} - \left(\frac{1}{3}(1)^3 - \frac{1}{1}\right) \\ &= \frac{8}{3} - \frac{1}{2} - \frac{1}{3} + 1 \\ &= \frac{17}{6}\end{aligned}$$

Remember that the evaluation is always done in the order of evaluation at the upper limit minus evaluation at the lower limit. Also be very careful with minus signs and parenthesis. It's very easy to forget them or mishandle them and get the wrong answer.

Notice as well that, in order to help with the evaluation, we rewrote the indefinite integral a little. In particular we got rid of the negative exponent on the second term.

(c) This integral is here to make a point. Recall that in order for us to do an integral the integrand must be continuous in the range of the limits. In this case the second term will have division by zero at $y=0$ and since $y=0$ is between the lower and upper limit this integrand is not continuous and so we can't do this integral.

Note that this problem will not prevent us from doing the integral in (b) since $y=0$ is not between the limits of integration in that case.

So what have we learned from this example?

First, in order to do a definite integral the first thing that we need to do is the indefinite integral. So we aren't going to get out of doing indefinite integrals, they will be in every integral that we'll be doing in the rest of this course.

Second, we need to be on the lookout for functions that aren't continuous at any point between the limits of integration. Also, it's important to note that this will only be a problem if the point(s) of discontinuity occur between the limits of integration. If the point of discontinuity occurs outside of the limits of integration the integral can still be evaluated.

Finally, note the difference between indefinite and definite integrals. Indefinite integrals are functions while definite integrals are numbers.

Let's work some more examples.

Example 2 Evaluate each of the following.

(a) $\int_{-3}^{1} 6x^2 - 5x + 2\,dx$

(b) $\int_{4}^{0} \sqrt{t}\left(t-2\right)dt$

(c) $\displaystyle\int_{1}^{2} \frac{2w^5 - w + 3}{w^2}\,dw$

(d) $\int_{25}^{-10} dR$

Solution

(a) There isn't a lot to this one other than simply doing the work.

$$\begin{aligned}\int_{-3}^{1} 6x^2 - 5x + 2\,dx &= \left(2x^3 - \frac{5}{2}x^2 + 2x\right)\Bigg|_{-3}^{1} \\ &= \left(2 - \frac{5}{2} + 2\right) - \left(-54 - \frac{45}{2} - 6\right) \\ &= 84\end{aligned}$$

(b) In this one we first need to multiply the integrand out before integrating, just as we did in the indefinite integral case.

$$\begin{aligned}\int_{4}^{0} \sqrt{t}\left(t-2\right)dt &= \int_{4}^{0} t^{\frac{3}{2}} - 2t^{\frac{1}{2}}\,dt \\ &= \left(\frac{2}{5}t^{\frac{5}{2}} - \frac{4}{3}t^{\frac{3}{2}}\right)\Bigg|_{4}^{0} \\ &= 0 - \left(\frac{2}{5}\left(4\right)^{\frac{5}{2}} - \frac{4}{3}\left(2\right)^{\frac{3}{2}}\right) \\ &= -\frac{32}{15}\end{aligned}$$

In the evaluation process recall that,

$$\left(4\right)^{\frac{5}{2}} = \left(\left(4\right)^{\frac{1}{2}}\right)^5 = \left(2\right)^5 = 32$$

$$(4)^{\frac{3}{2}} = \left((4)^{\frac{1}{2}}\right)^3 = (2)^3 = 8$$

Also, don't get excited about the fact that the lower limit of integration is larger than the upper limit of integration. That will happen on occasion.

(c) First, notice that we will have a division by zero issue at *w=0*, but since this isn't between the limits of integration we won't have to worry about it.

The first step that we'll need to do is break up the quotient so we can integrate.

$$\begin{aligned}\int_1^2 \frac{2w^5 - w + 3}{w^2}\,dw &= \int_1^2 2w^3 - \frac{1}{w} + 3w^{-2}\,dw \\ &= \left(\frac{1}{2}w^4 - \ln|w| - \frac{3}{w}\right)\Big|_1^2 \\ &= \left(8 - \ln 2 - \frac{3}{2}\right) - \left(\frac{1}{2} - \ln 1 - 3\right) \\ &= 9 - \ln 2\end{aligned}$$

Don't get excited about answers that don't come down to a simple integer or fraction. Often times they won't. Also don't forget that ln1=0.

(d) This one is actually pretty easy. Recall that we're just integrating 1!.

$$\begin{aligned}\int_{25}^{-10} dR &= R\Big|_{25}^{-10} \\ &= -10 - 25 \\ &= -35\end{aligned}$$

The last set of examples dealt exclusively with integrating powers of *x*. Let's work a couple of examples that involve other functions.

Example 3 Evaluate each of the following.

(a) $\int_0^1 4x - 6\sqrt[3]{x^2}\,dx$

(b) $\int_0^{\frac{\pi}{3}} 2\sin\theta - 5\cos\theta\,d\theta$

(c) $\int_{\pi/6}^{\pi/4} 5 - 2\sec z\tan z\,dz$

(d) $\int_{-20}^{-1} \frac{3}{\mathbf{e}^{-z}} - \frac{1}{3z}\,dz$

(e) $\int_{-2}^{3} 5t^6 - 10t + \frac{1}{t}\,dt$

Solution

(a) This one is here mostly here to contrast with the next example.

$$\begin{aligned}\int_0^1 4x - 6\sqrt[3]{x^2}\,dx &= \int_0^1 4x - 6x^{\frac{2}{3}}\,dx \\ &= \left(2x^2 - \frac{18}{5}x^{\frac{5}{3}}\right)\Bigg|_0^1 \\ &= 2 - \frac{18}{5} - (0) \\ &= -\frac{8}{5}\end{aligned}$$

(b) Be careful with signs with this one.

$$\begin{aligned}\int_0^{\frac{\pi}{3}} 2\sin\theta - 5\cos\theta\,d\theta &= \left(-2\cos\theta - 5\sin\theta\right)\Big|_0^{\pi/3} \\ &= -2\cos\left(\frac{\pi}{3}\right) - 5\sin\left(\frac{\pi}{3}\right) - \left(-2\cos 0 - 5\sin 0\right) \\ &= -1 - \frac{5\sqrt{3}}{2} + 2 \\ &= 1 - \frac{5\sqrt{3}}{2}\end{aligned}$$

Compare this answer to the previous answer, especially the evaluation at zero. It's very easy to get into the habit of just writing down zero when evaluating a function at zero. However, there are many functions out there that aren't zero when evaluated at zero so be careful.

(c) Not much to do other than do the integral.

$$\begin{aligned}\int_{\pi/6}^{\pi/4} 5 - 2\sec z\tan z\,dz &= \left(5z - 2\sec z\right)\Big|_{\pi/6}^{\pi/4} \\ &= 5\left(\frac{\pi}{4}\right) - 2\sec\left(\frac{\pi}{4}\right) - \left(5\left(\frac{\pi}{6}\right) - 2\sec\left(\frac{\pi}{6}\right)\right) \\ &= \frac{5\pi}{12} - 2\sqrt{2} + \frac{4}{\sqrt{3}}\end{aligned}$$

Recall that

$$\sec z = \frac{1}{\cos z}$$

and so if we can evaluate cosine at these angles we can evaluate secant at these angles.

(d) In order to do this one will need to rewrite both of the terms in the integral.

$$\int_{-20}^{-1} \frac{3}{\mathbf{e}^{-z}} - \frac{1}{3z}\,dz = \int_{-20}^{-1} 3\mathbf{e}^{z} - \frac{1}{3}\frac{1}{z}\,dz$$

Recall the following facts about exponents.

$$x^{-a} = \frac{1}{x^{a}} \qquad\qquad \frac{1}{x^{-a}} = x^{a}$$

Now the integral.

$$\begin{aligned}\int_{-20}^{-1} \frac{3}{\mathbf{e}^{-z}} - \frac{1}{3z}\,dz &= \left(3\mathbf{e}^{z} - \frac{1}{3}\ln|z|\right)\Bigg|_{-20}^{-1} \\ &= 3\mathbf{e}^{-1} - \frac{1}{3}\ln|-1| - \left(3\mathbf{e}^{-20} - \frac{1}{3}\ln|-20|\right) \\ &= 3\mathbf{e}^{-1} - 3\mathbf{e}^{-20} + \frac{1}{3}\ln|20|\end{aligned}$$

Just leave the answer like this. It's messy, but it's also exact.

Note that the absolute value bars on the logarithm are required here. Without them we couldn't have done the evaluation.

(e) This integral can't be done. There is division by zero in the third term at $t=0$ and $t=0$ does lie between the limits of integration. The fact that the first two terms can be integrated doesn't matter. If even one term in the integral can't be integrated then the whole integral can't be done.

So, we've seen some definite integrals computed. Remember that the vast majority of the work in computing them is first finding the indefinite integral. Once we've found that the rest is just some number crunching.

There are a couple of particularly tricky integrals that we need to take a look at next. Actually they are only tricky until you see how to do them, so don't get too excited about them.

Let's take a look at the first one. This one involves integrating a piecewise function.

Example 4 Given,

$$f(x) = \begin{cases} 6 & \text{if } x > 1 \\ 3x^{2} & \text{if } x \le 1 \end{cases}$$

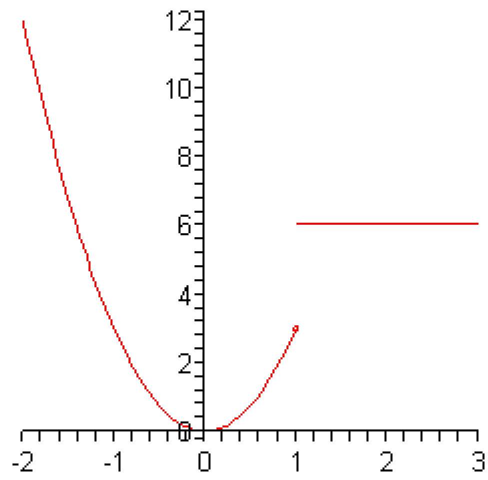
Evaluate each of the following integrals.

(a) $\int_{-2}^{3} f(x)\,dx$

(b) $\int_{10}^{22} f(x)\,dx$

Solution

Let's first start with a graph of this function.



The graph reveals a problem. This function is not continuous at *x=1* and we're going to have to watch out for that.

(a) In this first integral the point of discontinuity is between the limits of integration. However, unlike the previous examples where this meant the integral can't be done, we can still do this integral.

The previous examples where we had functions that weren't continuous we had division by zero and no matter how hard we try we can't get rid of that problem. With this integral however, all we need to do is remember Property 5 from the previous section. This property tells us that we can write the integral as follows,

$$\int_{-2}^{3} f(x)\,dx = \int_{-2}^{1} f(x)\,dx + \int_{1}^{3} f(x)\,dx$$

On each of these intervals the function is continuous. In fact we can say more. In the first integral we will have x between -2 and 1 and this means that we can use the second function for *f(x)* and likewise for the second integral x will be between 1 and 3 and so we can use the first function for *f(x)*. The integral in this case is,

$$\begin{aligned}\int_{-2}^{3} f(x)\,dx &= \int_{-2}^{1} f(x)\,dx + \int_{1}^{3} f(x)\,dx \\ &= \int_{-2}^{1} 3x^2\,dx + \int_{1}^{3} 6\,dx \\ &= x^3\Big|_{-2}^{1} + 6x\Big|_{1}^{3} \\ &= 1-(-8)+(18-6) \\ &= 21\end{aligned}$$

So, to integrate a piecewise function all we need to do is break up the integral at the break point and then integrate each piece.

(b) The limits for this integral lie entirely in the range for the first function and so there isn't anything to do. All we need to do is do the integral.

$$\begin{aligned}\int_{10}^{22} f(x)\,dx &= \int_{10}^{22} 6\,dx\\ &= 6x\Big|_{10}^{22}\\ &= 132-60\\ &= 72\end{aligned}$$

The next integral that we need to look at is how to integrate an absolute value function.

Example 5 Evaluate the following integral.

$$\int_0^3 \left|3t-5\right| dt$$

Solution

Recall that the point behind indefinite integration (which we'll need to do in this problem) is to determine what function we differentiated to get the integrand. To this point we've not seen any functions that will differentiate to get an absolute value.

The only way that we can do this problem is to get rid of the absolute value. To do this we need to recall the definition of absolute value.

$$|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$$

What we need to do is determine where the quantity on the inside is negative and where it is positive. It looks like if $t > \frac{5}{3}$ the quantity inside the absolute value is positive and if $t < \frac{5}{3}$ the quantity inside the absolute value is negative. Note as well that this point is between the limits of integration and so, if we break up the integral at this point we get,

$$\int_0^3 \left|3t-5\right| dt = \int_0^{\frac{5}{3}} \left|3t-5\right| dt + \int_{\frac{5}{3}}^3 \left|3t-5\right| dt$$

Now, in the first integrals we have $t < \frac{5}{3}$ and so we can drop the absolute value bars and put in a minus sign. Likewise in the second integral we have $t > \frac{5}{3}$ and so we can just drop the absolute value bars.

Doing this gives us,

$$\begin{aligned}\int_0^3 \left|3t-5\right| dt &= \int_0^{\frac{5}{3}} -\left(3t-5\right) dt + \int_{\frac{5}{3}}^{3} 3t-5\,dt \\ &= \int_0^{\frac{5}{3}} -3t+5\,dt + \int_{\frac{5}{3}}^{3} 3t-5\,dt \\ &= \left(-\frac{3}{2}t^2+5t\right)\Bigg|_0^{\frac{5}{3}} + \left(\frac{3}{2}t^2-5t\right)\Bigg|_{\frac{5}{3}}^{5} \\ &= -\frac{3}{2}\left(\frac{5}{3}\right)^2+5\left(\frac{5}{3}\right)-\left(0\right)+\left(\frac{3}{2}\left(5\right)^2-5\left(5\right)-\left(\frac{3}{2}\left(\frac{5}{3}\right)^2-5\left(\frac{5}{3}\right)\right)\right) \\ &= \frac{25}{6}+\frac{50}{3} \\ &= \frac{125}{6}\end{aligned}$$

Integrating absolute value functions isn't too bad. First, determine where the quantity is negative and positive and then break up the integral so that in each range of limits the quantity is only positive or negative. Once this is done we can drop the absolute value bars (with some negative signs also potentially showing up) and then we can do the integral as we've always done.

Substitution Rule for Definite Integrals

We now need to come back and revisit the substitution rule as it applies to definite integrals. At some level there really isn't a lot to do in this section. Recall that the first step in doing a definite integral is to compute the indefinite integral and that hasn't changed. We will still compute the indefinite integral first. This means that we already know how to do these. We use the substitution rule to find the indefinite integral and then do the evaluation.

There are however, two ways to work these problems. One of the ways of working these problems is the probably the most obvious at this point, but also has a point in the process where we can get in trouble if we aren't paying attention.

Let's work an example to see this.

Example 1 Evaluate the following definite integral.

$$\int_{-2}^{0} 2t^2\sqrt{1-4t^3}\,dt$$

Solution

As mentioned there are two solution methods. Let's take a look at the first method.

Solution 1 :
This first method is to do what we've been doing to this point. We first compute the indefinite integral using the substitution rule. Note however, that we will constantly remind ourselves that this is a definite integral by putting the limits on the integral at each step. Without the limits it's easy to forget that we had a definite integral at the end.

In this case the substitution is,

$$u = 1 - 4t^3 \qquad du = -12t^2 dt \qquad \Rightarrow \qquad t^2 dt = -\frac{1}{12} du$$

Plugging this into the integral gives,

$$\begin{aligned}\int_{-2}^{0} 2t^2 \sqrt{1-4t^3}\, dt &= -\frac{1}{6}\int_{-2}^{0} u^{\frac{1}{2}}\, du \\ &= -\frac{1}{9} u^{\frac{3}{2}}\Big|_{-2}^{0}\end{aligned}$$

Notice that we didn't do the evaluation yet. This is where the potential problem arises with this solution method. The limits given here are from the original integral and hence are values of *t*. We have *u*'s in our solution. We can't plug values of *t* in for *u*.

Therefore, we will have to go back to *t*'s before we do the substitution. This is the standard step in the substitution process, but it is often forgotten when doing definite integrals. Note as well that in this case, if we don't go back to *t*'s we will have a small problem in that one of the evaluations will end up giving us a complex number.

So, finishing this problem gives,

$$\begin{aligned}\int_{-2}^{0} 2t^2 \sqrt{1-4t^3}\, dt &= -\frac{1}{9}\left(1-4t^3\right)^{\frac{3}{2}}\Big|_{-2}^{0} \\ &= -\frac{1}{9} - \left(-\frac{1}{9}\left(33\right)^{\frac{3}{2}}\right) \\ &= \frac{1}{9}\left(33\sqrt{33} - 1\right)\end{aligned}$$

So, that was the first solution method. Let's take a look at the second method.

Solution 2 :
Note that this solution method isn't really all that different from the first method. In this method we are going to remember that when doing a substitution we want to eliminate all the *t*'s in the integral and write everything in terms of *u*.

In this method we are going to recall something we pointed out in the first method. The limits are *t*'s and we've got an equation that tells us what *u* is in terms of *t* (our substitution) and so we can also convert the limits to *u*'s.

In this case here is the substitution as well as the limit conversions.

$$u = 1 - 4t^3 \qquad du = -12t^2 dt \qquad \Rightarrow \qquad t^2 dt = -\frac{1}{12} du$$

$$t = -2 \qquad \Rightarrow \qquad u = 1 - 4(-2)^3 = 33$$

$$t = 0 \qquad \Rightarrow \qquad u = 1 - 4(0)^3 = 1$$

The integral is now,

$$\begin{aligned}\int_{-2}^{0} 2t^2 \sqrt{1-4t^3}\, dt &= -\frac{1}{6}\int_{33}^{1} u^{\frac{1}{2}}\, du \\ &= -\frac{1}{9} u^{\frac{3}{2}}\Big|_{33}^{1} \\ &= -\frac{1}{9} - \left(-\frac{1}{9}(33)^{\frac{3}{2}}\right) \\ &= \frac{1}{9}\left(33\sqrt{33} - 1\right)\end{aligned}$$

We got exactly the same answer and this time didn't have to worry about going back to t's in our answer.

So, we've seen two solution techniques for computing definite integrals that require the substitution rule. Both are valid solution methods. We will be using the second exclusively however since it make the evaluation step a little easier.

Let's work some more examples.

Example 2 Evaluate each of the following.

(a) $\int_{-1}^{5} (1+w)\left(2w+w^2\right)^5 dw$

(b) $\int_{-2}^{-6} \frac{4}{(1+2x)^3} - \frac{5}{1+2x}\, dx$

(c) $\int_{-5}^{5} \frac{4t}{2-8t^2}\, dt$

(d) $\int_{0}^{\frac{1}{2}} \mathbf{e}^y + 2\cos(\pi y)\, dy$

(e) $\int_{\frac{\pi}{3}}^{0} 3\sin\left(\frac{z}{2}\right) - 5\cos(\pi - z)\, dz$

Solution

Since we've done quite a few substitution rule integrals to this time we aren't going to put a lot of effort into the substitution part of things here.

(a) The substitution here is,

$$u = 2w + w^2 \qquad du = (2+2w)\,dw \qquad \Rightarrow \qquad (1+w)\,dw = \frac{1}{2}du$$

$$w = -1 \qquad \Rightarrow \qquad u = -1$$
$$w = 5 \qquad \Rightarrow \qquad u = 35$$

Sometimes a limit will remain the same after the substitution. Don't get excited when it happens and don't expect it to happen all the time.

Here is the integral,

$$\begin{aligned}\int_{-1}^{5}(1+w)\left(2w+w^2\right)^5 dw &= \frac{1}{2}\int_{-1}^{35}u^5\,du \\ &= \left.\frac{1}{12}u^6\right|_{-1}^{35} \\ &= 153188802\end{aligned}$$

Don't get excited about large numbers for answers here. Sometime they are. That's life.

(b) Here is the substitution,

$$u = 1+2x \qquad du = 2dx \qquad \Rightarrow \qquad dx = \frac{1}{2}du$$

$$x = -2 \qquad \Rightarrow \qquad u = -3$$
$$x = -6 \qquad \Rightarrow \qquad u = -11$$

The integral is then,

$$\begin{aligned}\int_{-2}^{-6}\frac{4}{(1+2x)^3}-\frac{5}{1+2x}\,dx &= \frac{1}{2}\int_{-3}^{-11}4u^{-3}-\frac{5}{u}\,dx \\ &= \left.\frac{1}{2}\left(-2u^{-2}-5\ln|u|\right)\right|_{-3}^{-11} \\ &= \frac{1}{2}\left(-\frac{2}{121}-5\ln 11\right)-\frac{1}{2}\left(-\frac{2}{9}-5\ln 3\right) \\ &= \frac{112}{1089}-\frac{5}{2}\ln 11+\frac{5}{2}\ln 3\end{aligned}$$

(c) Be careful with this integral. The denominator is zero at $t = \pm\frac{1}{2}$ and both of these are between the limits of integration. Therefore, this integrand is not continuous in the interval and so the integral can't be done.

(d) This integral needs to be split into two integrals since the first term doesn't require a substitution and the second does.

$$\int_0^{\frac{1}{2}} \mathbf{e}^y + 2\cos(\pi y)\,dy = \int_0^{\frac{1}{2}} \mathbf{e}^y\,dy + \int_0^{\frac{1}{2}} 2\cos(\pi y)\,dy$$

Here is the substitution for the second term.

$$u = \pi y \qquad du = \pi dy \qquad \Rightarrow \qquad dy = \frac{1}{\pi}du$$

$$y = 0 \qquad \Rightarrow \qquad u = 0$$

$$y = \frac{1}{2} \qquad \Rightarrow \qquad u = \frac{\pi}{2}$$

Here is the integral.

$$\begin{aligned}\int_0^{\frac{1}{2}} \mathbf{e}^y + 2\cos(\pi y)\,dy &= \int_0^{\frac{1}{2}} \mathbf{e}^y\,dy + \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos(u)\,du \\ &= \mathbf{e}^y\Big|_0^{\frac{1}{2}} + \frac{2}{\pi}\sin u\Big|_0^{\frac{\pi}{2}} \\ &= \mathbf{e}^{\frac{1}{2}} - \mathbf{e}^0 + \frac{2}{\pi}\sin\frac{\pi}{2} - \frac{2}{\pi}\sin 0 \\ &= \mathbf{e}^{\frac{1}{2}} - 1 + \frac{2}{\pi}\end{aligned}$$

(e) This integral will require two substitutions. So first split up the integral.

$$\int_{\frac{\pi}{3}}^{0} 3\sin\left(\frac{z}{2}\right) - 5\cos(\pi - z)\,dz = \int_{\frac{\pi}{3}}^{0} 3\sin\left(\frac{z}{2}\right)dz - \int_{\frac{\pi}{3}}^{0} 5\cos(\pi - z)\,dz$$

There are the two substitutions for these integrals.

$$u = \frac{z}{2} \qquad du = \frac{1}{2}dz \qquad \Rightarrow \qquad dz = 2du$$

$$z = \frac{\pi}{3} \qquad \Rightarrow \qquad u = \frac{\pi}{6}$$

$$z = 0 \qquad \Rightarrow \qquad u = 0$$

$$v = \pi - z \qquad dv = -dz \qquad \Rightarrow \qquad dz = -dv$$

$$z = \frac{\pi}{3} \qquad \Rightarrow \qquad v = \frac{2\pi}{3}$$

$$z = 0 \qquad \Rightarrow \qquad v = \pi$$

Here is the integral,

$$\int_{\frac{\pi}{3}}^{0} 3\sin\left(\frac{z}{2}\right) - 5\cos\left(\pi - z\right)dz = 6\int_{\frac{\pi}{6}}^{0}\sin\left(u\right)du + 5\int_{\frac{2\pi}{3}}^{\pi}\cos\left(v\right)dv$$

$$= -6\cos\left(u\right)\Big|_{\frac{\pi}{6}}^{0} + 5\sin\left(v\right)\Big|_{\frac{2\pi}{3}}^{\pi}$$

$$= 3\sqrt{3} - 6 + \left(-\frac{5\sqrt{3}}{2}\right)$$

$$= \frac{\sqrt{3}}{2} - 6$$

Let's work another set of examples. These are a little tougher (at least in appearance) than the previous set.

Example 3 Evaluate each of the following.

(a) $\int_{0}^{\ln(1-\pi)} \mathbf{e}^{x}\cos\left(1-\mathbf{e}^{x}\right)dx$

(b) $\displaystyle\int_{\mathbf{e}^2}^{\mathbf{e}^6} \frac{\left[\ln t\right]^4}{t}\,dt$

(c) $\displaystyle\int_{\frac{\pi}{12}}^{\frac{\pi}{9}} \frac{\sec\left(3P\right)\tan\left(3P\right)}{\sqrt[3]{2+\sec\left(3P\right)}}\,dP$

(d) $\int_{-\pi}^{\frac{\pi}{2}} \cos\left(x\right)\cos\left(\sin\left(x\right)\right)dx$

(e) $\displaystyle\int_{\frac{1}{50}}^{2} \frac{\mathbf{e}^{\frac{2}{w}}}{w^2}\,dw$

Solution

(a) The limits are a little unusual in this case, but that will happen sometimes with substitution rule on definite integrals. Here is the substitution.

$$u = 1 - \mathbf{e}^{x} \qquad\qquad du = -\mathbf{e}^{x}dx$$

$$x = 0 \qquad \Rightarrow \qquad u = 1 - \mathbf{e}^{0} = 1 - 1 = 0$$

$$x = \ln\left(1-\pi\right) \qquad \Rightarrow \qquad u = 1 - \mathbf{e}^{\ln(1-\pi)} = 1 - \left(1-\pi\right) = \pi$$

The integral is then,

$$\int_{0}^{\ln(1-\pi)} \mathbf{e}^{x}\cos\left(1-\mathbf{e}^{x}\right)dx = -\int_{0}^{\pi}\cos u\,du$$

$$= -\sin\left(u\right)\Big|_{0}^{\pi}$$

$$= -\left(\sin\pi - \sin 0\right)$$

$$= 0$$

(b) Here is the substitution for this problem.

$$u = \ln t \qquad du = \frac{1}{t}\,dt$$

$$t = \mathbf{e}^2 \qquad \Rightarrow \qquad u = \ln \mathbf{e}^2 = 2$$

$$t = \mathbf{e}^6 \qquad \Rightarrow \qquad u = \ln \mathbf{e}^6 = 6$$

The integral is,

$$\begin{aligned}\int_{\mathbf{e}^2}^{\mathbf{e}^6} \frac{[\ln t]^4}{t}\,dt &= \int_2^6 u^4\,du \\ &= \left.\frac{1}{5}u^5\right|_2^6 \\ &= \frac{7744}{5}\end{aligned}$$

(c) Here is the substitution for this problem.

$$u = 2 + \sec(3P) \qquad du = 3\sec(3P)\tan(3P)\,dP \quad \Rightarrow \quad \sec(3P)\tan(3P)\,dP = \frac{1}{3}du$$

$$P = \frac{\pi}{12} \qquad \Rightarrow \qquad u = 2 + \sec\left(\frac{\pi}{4}\right) = 2 + \sqrt{2}$$

$$P = \frac{\pi}{9} \qquad \Rightarrow \qquad u = 2 + \sec\left(\frac{\pi}{3}\right) = 4$$

This is a somewhat messy substitution, but sometimes they are. Here is the integral,

$$\begin{aligned}\int_{\frac{\pi}{12}}^{\frac{\pi}{9}} \frac{\sec(3P)\tan(3P)}{\sqrt[3]{2+\sec(3P)}}\,dP &= \frac{1}{3}\int_{2+\sqrt{2}}^{4} u^{-\frac{1}{3}}\,du \\ &= \left.\frac{1}{2}u^{\frac{2}{3}}\right|_{2+\sqrt{2}}^{4} \\ &= \frac{1}{2}\left(4^{\frac{3}{2}} - \left(2+\sqrt{2}\right)^{\frac{2}{3}}\right) \\ &= \frac{1}{2}\left(8 - \left(2+\sqrt{2}\right)^{\frac{2}{3}}\right)\end{aligned}$$

Also a messy answer, but again that's life on occasion.

(d) This problem not as bad as it looks. Here is the substitution.

$$u = \sin x \qquad du = \cos x\,dx$$
$$x = \frac{\pi}{2} \quad \Rightarrow \quad u = \sin\frac{\pi}{2} = 1$$
$$x = -\pi \quad \Rightarrow \quad u = \sin\left(-\pi\right) = 0$$

The cosine in the front will get substituted away in the differential. Here is the integral.

$$\begin{aligned}\int_{-\pi}^{\frac{\pi}{2}} \cos\left(x\right)\cos\left(\sin\left(x\right)\right)dx &= \int_0^1 \cos u\,du \\ &= \sin\left(u\right)\Big|_0^1 \\ &= \sin\left(1\right) - \sin\left(0\right) \\ &= \sin\left(1\right)\end{aligned}$$

Don’t get excited about these kinds of answers. On occasion we will end up with trig function evaluations like this.

(e) This is also a tricky substitution (at least until you see it). Here it is,

$$u = \frac{2}{w} \qquad du = -\frac{2}{w^2}\,dw \qquad \Rightarrow \qquad \frac{1}{w^2}\,dw = -\frac{1}{2}\,du$$
$$w = 2 \quad \Rightarrow \quad u = 1$$
$$w = \frac{1}{50} \quad \Rightarrow \quad u = 100$$

Here is the integral.

$$\begin{aligned}\int_{\frac{1}{50}}^{2} \frac{\mathbf{e}^{\frac{2}{w}}}{w^2}\,dw &= -\frac{1}{2}\int_{100}^{1} \mathbf{e}^u\,du \\ &= -\frac{1}{2}\mathbf{e}^u\Big|_{100}^{1} \\ &= -\frac{1}{2}\left(\mathbf{e}^1 - \mathbf{e}^{100}\right)\end{aligned}$$

In this last set of examples we saw some tricky substitutions and messy limits, but these are a fact of life with some substitution problems and so we need to be prepared for dealing with them when the happen.

Even and Odd Functions

This is the last topic that we need to discuss in this chapter. It is probably better suited in the previous section, but that section has already gotten fairly large so I decided to put it here.

First, recall that an even function is any function which satisfies,

$$f(-x) = f(x)$$

Typical examples of even functions are,

$$f(x) = x^2 \qquad\qquad f(x) = \cos(x)$$

An odd function is any function which satisfies,

$$f(-x) = -f(x)$$

The typical examples of odd functions are,

$$f(x) = x^3 \qquad\qquad f(x) = \sin(x)$$

There are a couple of nice facts about integrating even and odd functions over the interval [-*a*,*a*]. If *f(x)* is an even function then,

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

Likewise, if *f(x)* is an odd function then,

$$\int_{-a}^{a} f(x)\,dx = 0$$

Note that in order to use these facts the limit of integration must be the same number, but opposite signs!

Example 4 Integrate each of the following.

(a) $\int_{-2}^{2} 4x^4 - x^2 + 1\,dx$

(b) $\int_{-10}^{10} x^5 + \sin(x)\,dx$

Solution

Neither of these are terribly difficult integrals, but we can use the facts on them anyway.

(a) In this case the integrand is even and the interval is correct so,

$$\begin{aligned}\int_{-2}^{2} 4x^4 - x^2 + 1\,dx &= 2\int_{0}^{2} 4x^4 - x^2 + 1\,dx \\ &= 2\left(\frac{4}{5}x^5 - \frac{1}{3}x^3 + x\right)\Bigg|_0^2 \\ &= \frac{748}{15}\end{aligned}$$

So, using the fact cut the evaluation in half (in essence since one of the new limits was zero).

(b) The integrand in this case is odd and the interval is in the correct form and so we don’t even need to integrate. Just use the fact.

$$\int_{-10}^{10} x^5 + \sin(x)\,dx = 0$$

Note that the limits of integration are important here. Take the last integral as an example. A small change to the limits will not give us zero.

$$\int_{-10}^{9} x^5 + \sin(x)\,dx = \cos(10) - \cos(9) - \frac{468559}{6} = -78093.09461$$

The moral here is to be careful and not misuse these facts.

Applications of Integrals

Introduction

In this last chapter of this course we will be taking a look at a couple of applications of integrals. There are many other applications, however many of them require integration techniques that are typically taught in Calculus II. We will therefore be focusing on applications that can be done only with knowledge taught in this course.

Because this chapter is focused on the applications of integrals it is assumed in all the examples that you are capable of doing the integrals. There will not be as much detail in the integration process in the examples in this chapter as there was in the examples in the previous chapter.

Here is a listing of applications covered in this chapter.

Average Function Value – We can use integrals to determine the average value of a function.

Area Between Two Curves – In this section we'll take a look at determining the area between two curves.

Volumes of Solids of Revolution / Method of Rings – This is the first of two sections devoted to find the volume of a solid of revolution. In this section we look that the method of rings/disks.

Volumes of Solids of Revolution / Method of Cylinders – This is the second section devoted to finding the volume of a solid of revolution. Here we will look at the method of cylinders.

Work – The final application we will look at is determining the amount of work required to move an object.

Average Function Value

The first application of integrals that we'll take a look at is the average value of a function *f(x)* over the interval [*a*,*b*] is given by,

$$f_{avg} = \frac{1}{b-a}\int_a^b f\left(x\right)dx$$

Example 1 Determine the average value of each of the following functions on the given interval.

(a) $f\left(t\right) = t^2 - 5t + 6\cos\left(\pi t\right)$ on $\left[-1, \frac{5}{2}\right]$

(b) $R\left(z\right) = \sin\left(2z\right)\mathbf{e}^{1-\cos(2z)}$ on $\left[-\pi, \pi\right]$

Solution

(a) There's really not a whole lot to do in this problem other than just use the formula.

$$\begin{aligned} f_{avg} &= \frac{1}{\frac{5}{2}-\left(-1\right)}\int_{-1}^{\frac{5}{2}} t^2 - 5t + 6\cos\left(\pi t\right)dt \\ &= \frac{2}{7}\left(\frac{1}{3}t^3 - \frac{5}{2}t^2 + \frac{6}{\pi}\sin\left(\pi t\right)\right)\Bigg|_{-1}^{\frac{5}{2}} \\ &= \frac{12}{7\pi} - \frac{13}{6} \\ &= -1.620993 \end{aligned}$$

You caught the substitution needed for the third term right?

So, the average value of this function of the given interval is -1.620993.

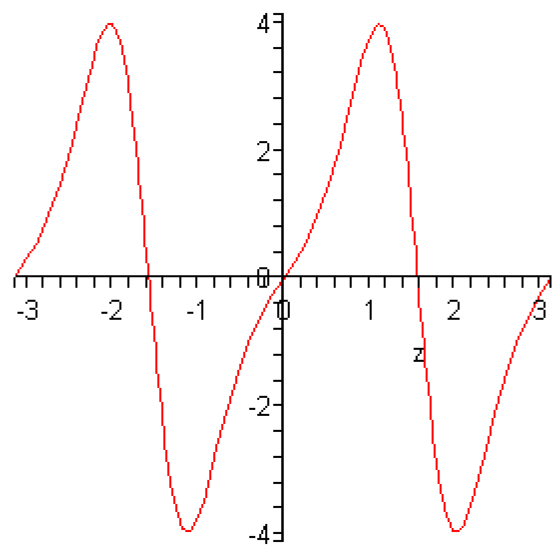
(b) Again, not much to do here other than use the formula. Note that the integral will need the following substitution.

$$u = 1 - \cos\left(2z\right)$$

Here is the average value of this function,

$$\begin{aligned} R_{avg} &= \frac{1}{\pi - \left(-\pi\right)}\int_{\pi}^{-\pi} \sin\left(2z\right)\mathbf{e}^{1-\cos(2z)}\,dz \\ &= \frac{1}{2}\mathbf{e}^{1-\cos(2z)}\Big|_{-\pi}^{\pi} \\ &= 0 \end{aligned}$$

So, in this case the average function value is zero. Do not get excited about getting zero here. It will happen on occasion. In fact, if you look at the graph of the function on this interval it's not too hard to see that this is the correct answer.



There is also a theorem that gives a similar result.

The Mean Value Theorem for Integrals

If *f(x)* is a continuous function on [*a*,*b*] then there is a number *c* in [*a*,*b*] such that,

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

Note that this is very similar to the Mean Value Theorem that we saw in the Derivatives Applications chapter.

Example 2 Determine the number *c* that satisfies the Mean Value Theorem for Integrals for the function $f(x) = x^2 + 3x + 2$ on the interval [1,4]

Solution

First let's notice that the function is a polynomial and so is continuous on the given interval. This means that we can use the Mean Value Theorem. So, let's do that.

$$\int_1^4 x^2 + 3x + 2\,dx = \left(c^2 + 3c + 2\right)(4-1)$$

$$\left.\left(\frac{1}{3}x^2 + \frac{3}{2}x^2 + 2x\right)\right|_1^4 = 3\left(c^2 + 3c + 2\right)$$

$$\frac{99}{2} = 3c^2 + 9c + 6$$

$$0 = 3c^2 + 9c - \frac{87}{2}$$

This is a quadratic equation that we can solve. Using the quadratic formula we get the following two solutions,

$$x = \frac{-3+\sqrt{67}}{2} = 2.593$$
$$x = \frac{-3-\sqrt{67}}{2} = -5.593$$

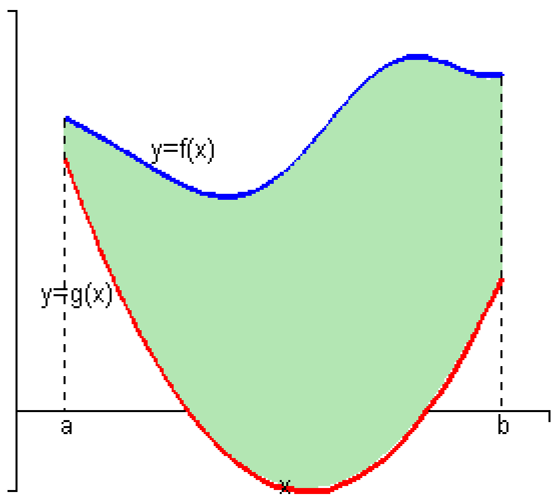
Clearly the second number is not in the interval and so that isn't the one that we're after. The first however is in the interval and so that's the number we want.

Note that it is possible for both numbers to be in the interval so don't expect only one to be in the interval.

Area Between Curves

In this section we are going to look at finding the area between two curves. There are actually two cases that we are going to be looking at.

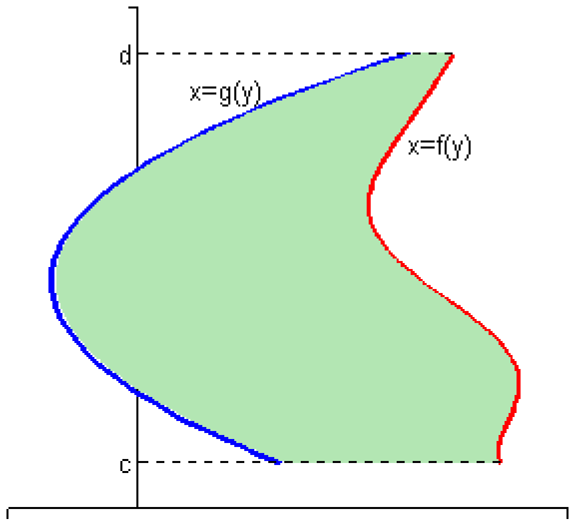
In the first case we are want to determine the area between $y=f(x)$ and $y=g(x)$ on the interval [a,b]. We are also going to assume that $f(x) \geq g(x)$.



In the final section of the Extras chapter we derived the following formula for the area in this case.

$$A = \int_a^b f(x) - g(x)\, dx \qquad \textbf{(1)}$$

The second case is almost identical to the first case. Here we are going to determine the area between *x=f(y)* and *x=g(y)* on the interval [*c*,*d*] with $f\left(y\right)\geq g\left(y\right)$.



In this case the formula is,

$$A=\int_{c}^{d} f\left(y\right)-g\left(y\right)dy \qquad \textbf{(2)}$$

Now (1) and (2) are perfectly serviceable formulas, however, it is sometimes easy to forget that these always require the first function to be the larger of the two functions. So, instead of these formulas we will instead use the following formulas.

In the first case we will use,

$$A=\int_{a}^{b}\left(\begin{array}{c}\text{upper}\\ \text{function}\end{array}\right)-\left(\begin{array}{c}\text{lower}\\ \text{function}\end{array}\right)dx, \qquad a\leq x\leq b \qquad \textbf{(3)}$$

In the second case we will use,

$$A=\int_{c}^{d}\left(\begin{array}{c}\text{right}\\ \text{function}\end{array}\right)-\left(\begin{array}{c}\text{left}\\ \text{function}\end{array}\right)dy, \qquad c\leq y\leq d \qquad \textbf{(4)}$$

Using these formulas will always force us to think about what is going on with each problem.

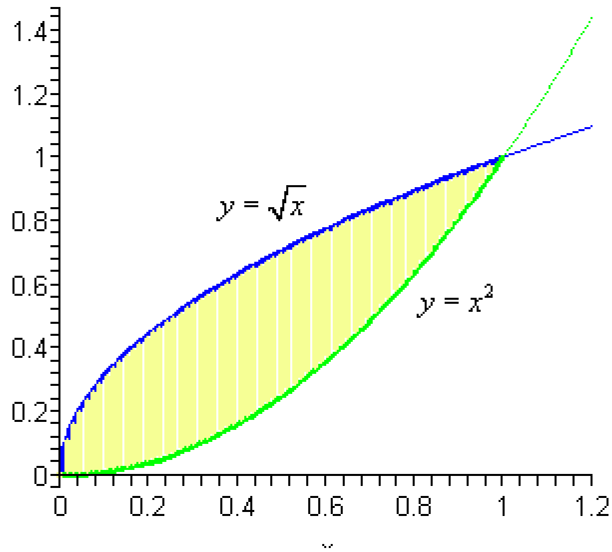
Let's work an example.

Example 1 Determine the area of the region enclosed by $y=x^2$ and $y=\sqrt{x}$.

Solution
First of all, just what do we mean by "area enclosed by". This means that the region must have one of the two curves on every boundary of the region.

So, to determine what this region is, it will be best to graph the two functions. Also notice that we are going to have to determine which of the two functions is the larger of the two and if you think about it that is not clear with these two functions.

So, here is a graph of the two functions with the enclosed region shaded.



The enclosed region is the yellow shaded region in the graph. Note that we don't take any part of the region to the right of the intersection point of these two graphs. In this region there is no boundary on the right side and so is not part of the enclosed area. Remember that one of the given functions must be on the each boundary of the enclosed region.

Also from this graph it's clear that the upper function will be dependent on the range of *x*'s that we use. Because of this you should always sketch of a graph of the region. Without a sketch it's often easy to mistake which of the two functions is the larger. In this case most would probably say that $y = x^2$ is the upper function and they would be right for the vast majority of the *x*'s. However, in this case it is the lower of the two functions.

The limits of integration for this will be the intersection points of the two curves. In this case it's pretty easy to see that they will intersect at $x=0$ and $x=1$ so these are the limits of integration.

So, the integral that we'll need to compute to find the area is,

$$\begin{aligned} A &= \int_a^b \begin{pmatrix} \text{upper} \\ \text{function} \end{pmatrix} - \begin{pmatrix} \text{lower} \\ \text{function} \end{pmatrix} dx \\ &= \int_0^1 \sqrt{x} - x^2 \, dx \\ &= \left(\frac{2}{3} x^{\frac{3}{2}} - \frac{1}{3} x^3 \right) \Bigg|_0^1 \\ &= \frac{1}{3} \end{aligned}$$

Before moving on to the next example, there are a couple of important things to note.

First, in almost all of these problems a graph is almost required. Often the bounding region, which will give the limits of integration, is difficult to determine without a graph.

Also, it can often be difficult to determine which of the functions is the upper function and with is the lower function without a graph. This is especially true in cases like the last example where the answer to that question actually depended upon the range of *x*'s that we were using.

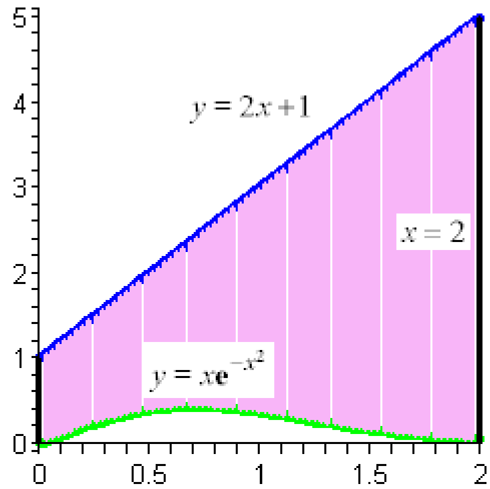
Finally, unlike the area under a curve that we looked at in the previous chapter the area between two curves will always be positive. If we get a negative number or zero we can be sure that we've made a mistake somewhere and will need to go back and find it.

Note as well that sometimes instead of saying region enclosed by we will say region bounded by. They mean the same thing.

Let's work some more examples.

Example 2 Determine the area of the region enclosed by $y = x\mathbf{e}^{-x^2}$, $y = 2x+1$, $x = 2$, and the *y*-axis.

Solution
In this case the last two pieces of information ($x = 2$ and the *y*-axis) tell us where to stop the region. Here is the graph with the enclosed region shaded in.



Here, unlike the first example, the two curves don't meet. Instead we rely on two vertical lines to bound the left and right sides of the region.

Here is the integral that will give the area.

$$
\begin{aligned}
A &= \int_a^b \begin{pmatrix}\text{upper}\\ \text{function}\end{pmatrix} - \begin{pmatrix}\text{lower}\\ \text{function}\end{pmatrix} dx \\
&= \int_0^2 2x + 1 - x\mathbf{e}^{-x^2}\, dx \\
&= \left(x^2 + x + \frac{1}{2}\mathbf{e}^{-x^2} \right)\Bigg|_0^2 \\
&= \frac{11}{2} + \frac{\mathbf{e}^{-4}}{2} = 5.509158
\end{aligned}
$$

Example 3 Determine the area of the region bounded by $y = 2x^2 + 10$ and $y = 4x + 16$.

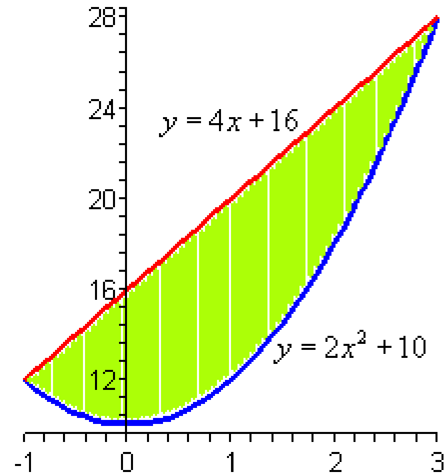
Solution

In this case the intersection points (which we'll need eventually) are not going to be easily found from the graph so let's go a head and get them now. In this case we can get the intersection points by setting the two equations equal.

$$
\begin{aligned}
2x^2 + 10 &= 4x + 16 \\
2x^2 - 4x - 6 &= 0 \\
2(x+1)(x-3) &= 0
\end{aligned}
$$

So it looks like the two curves will intersect at *x=-1* and *x=3.* If we need them we can get the *y* values corresponding to each of these by plugging the values back into either of the equations. Note that if you aren't good at graphing these points can help somewhat in getting the graph. We now know two points that are on the graph (-1,12) and (3,28). These will give us a place to start the graph.

Here is a graph of the region.



With the graph we can now identify the upper and lower function and so we can now find the enclosed area.

$$
\begin{aligned}
A &= \int_a^b \left(\begin{array}{c}\text{upper}\\\text{function}\end{array}\right) - \left(\begin{array}{c}\text{lower}\\\text{function}\end{array}\right) dx \\
&= \int_{-1}^{3} 4x + 16 - \left(2x^2 + 10\right) dx \\
&= \int_{-1}^{3} -2x^2 + 4x + 6 \, dx \\
&= \left(-\frac{2}{3}x^3 + 2x^2 + 6x\right)\Bigg|_{-1}^{3} \\
&= \frac{64}{3}
\end{aligned}
$$

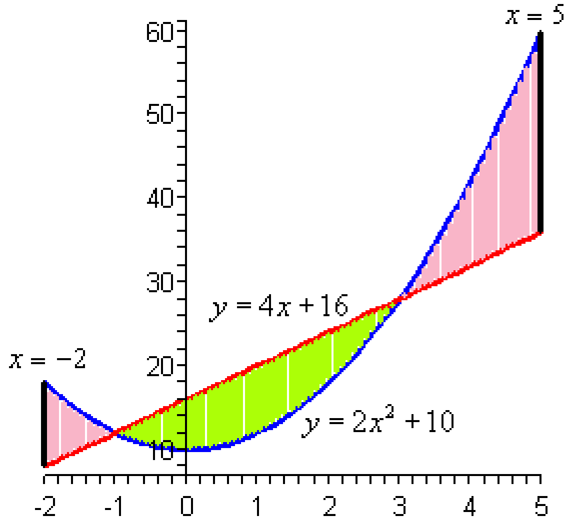
Be careful with parenthesis in these problems. One of the more common mistakes students make with these problems is to neglect parenthesis on the second term.

Example 4 Determine the area of the region bounded by $y = 2x^2 + 10$, $y = 4x + 16$, $x = -2$ and $x = 5$

Solution

So, the functions used in this problem are identical to the functions from the first problem. The difference is that we've extended the bounded region out from the intersection points.

Here is a graph of this region.



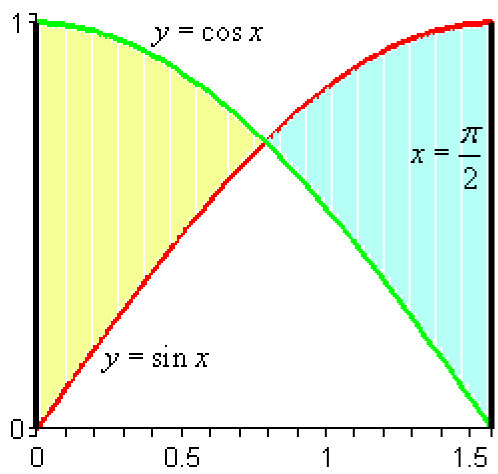
Okay, we have a small problem here. Our formula requires that one function always be the upper function and the other function always be the lower function. The actually isn't the problem that it might at first appear to be. All that we need to do is find the area of each of the three regions, which we can do, and then add them all up.

Here is the area.

$$\begin{aligned} A &= \int_{-2}^{-1} 2x^2+10-(4x+16)\,dx + \int_{-1}^{3} 4x+16-(2x^2+10)\,dx \int_{3}^{5} 2x^2+10-(4x+16)\,dx \\ &= \int_{-2}^{-1} 2x^2-4x-6\,dx + \int_{-1}^{3} -2x^2+4x+6\,dx + \int_{3}^{5} 2x^2-4x-6\,dx \\ &= \left(\frac{2}{3}x^3-2x^2-6x\right)\Bigg|_{-2}^{-1} + \left(-\frac{2}{3}x^3+2x^2+6x\right)\Bigg|_{-1}^{3} + \left(\frac{2}{3}x^3-2x^2-6x\right)\Bigg|_{3}^{5} \\ &= \frac{14}{3}+\frac{64}{3}+\frac{64}{3} \\ &= \frac{142}{3} \end{aligned}$$

Example 5 Determine the area of the region enclosed by $y=\sin x$, $y=\cos x$, $x=\dfrac{\pi}{2}$, and the y-axis.

Solution
First let's get a graph of the region.



So, we have another situation where we will need to do two integrals to get the area. The intersection point will be where

$$\sin x = \cos x$$

in the interval. This will be $x = \frac{\pi}{4}$. So, here is the area.

$$\begin{aligned} A &= \int_0^{\frac{\pi}{4}} \cos x - \sin x \, dx + \int_{\pi/4}^{\pi/2} \sin x - \cos x \, dx \\ &= \left(\sin x + \cos x\right)\Big|_0^{\frac{\pi}{4}} + \left(-\cos x - \sin x\right)\Big|_{\pi/4}^{\pi/2} \\ &= \sqrt{2} - 1 + \left(\sqrt{2} - 1\right) \\ &= 2\sqrt{2} - 2 = 0.828427 \end{aligned}$$

We will need to be careful with this next example.

Example 6 Determine the area of the region enclosed by $x = \frac{1}{2}y^2 - 3$ and $y = x - 1$.

Solution

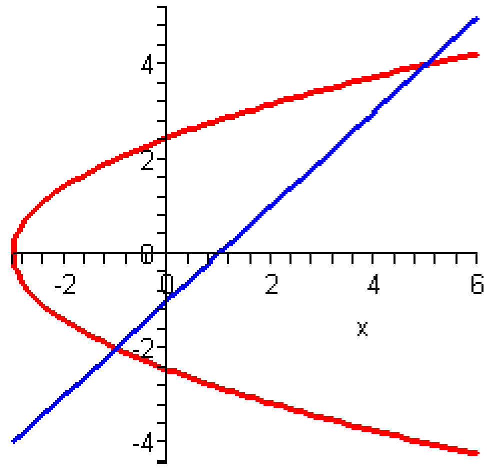
Don't let the first equation get you upset. We will have to deal with these kinds of equations occasionally so we'll need to get used to dealing with them.

First we will need intersection points for the two curves. In this case we'll get the intersection points by solving the second equation for x and the setting them equal.

$$\begin{aligned} y + 1 &= \frac{1}{2}y^2 - 3 \\ 2y + 2 &= y^2 - 6 \\ 0 &= y^2 - 2y - 8 \\ 0 &= \left(y - 4\right)\left(y + 2\right) \end{aligned}$$

So, it looks like the two curves will intersect at $y=-2$ and $y=4$ or if we need the full coordinates they will be : (-1,-2) and (5,4).

Here is a sketch of the two curves.

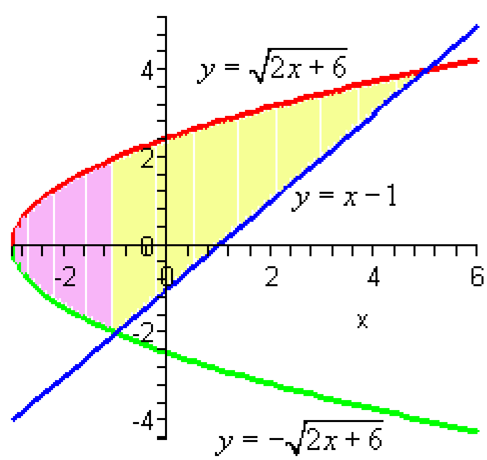


Now, we have a serious problem at this point if we aren't careful. To this point we've been using an upper function and a lower function. To do that here notice that there are actually two portions of the region that will have different lower functions. In the range [-2,-1] the parabola is actually both the upper and the lower function.

To use the formula that we've been using to this point we need to solve the parabola for y. This gives,

$$y = \pm\sqrt{2x+6}$$

where the "+" gives the upper portion of the parabola and the "-" gives the lower portion. Here is a sketch of the region if we were going to use the first formula.



The integrals for the area would then be,

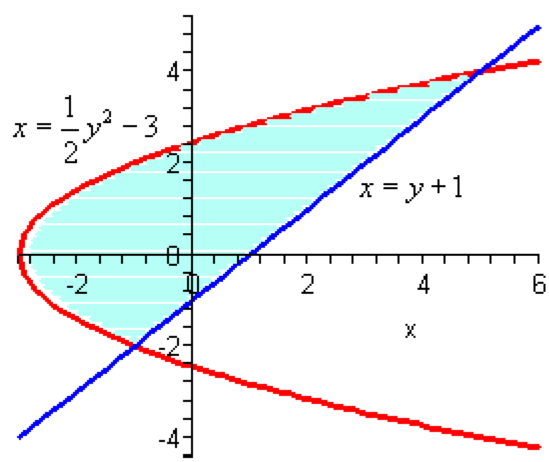
$$\begin{aligned}
A &= \int_{-3}^{-1} \sqrt{2x+6} - \left(-\sqrt{2x+6}\right) dx + \int_{-1}^{5} \sqrt{2x+6} - \left(x-1\right) dx \\
&= \int_{-3}^{-1} 2\sqrt{2x+6}\, dx + \int_{-1}^{5} \sqrt{2x+6} - x + 1\, dx \\
&= \int_{-3}^{-1} 2\sqrt{2x+6}\, dx + \int_{-1}^{5} \sqrt{2x+6}\, dx + \int_{-1}^{5} -x + 1\, dx \\
&= \frac{2}{3} u^{\frac{3}{2}} \Big|_{0}^{4} + \frac{1}{3} u^{\frac{3}{2}} \Big|_{4}^{16} + \left(-\frac{1}{2}x^2 + x\right)\Bigg|_{-1}^{5} \\
&= 18
\end{aligned}$$

While these integrals aren't terribly difficult they are more difficult than they need to be.

Recall that there is another formula for determining the area. It is,

$$A = \int_{c}^{d} \begin{pmatrix} \text{right} \\ \text{function} \end{pmatrix} - \begin{pmatrix} \text{left} \\ \text{function} \end{pmatrix} dy, \qquad\qquad c \le y \le d$$

and in our case we do have one function that is always on the left and the other is always on the right. So, in this case this is definitely the way to go. Note that we will need to rewrite the equation of the line since it will need to be in the form $x = f\left(y\right)$ but that is easy enough to do. Here is the graph for using this formula.



In this case the area is,

$$
\begin{aligned}
A &= \int_c^d \begin{pmatrix} \text{right} \\ \text{function} \end{pmatrix} - \begin{pmatrix} \text{left} \\ \text{function} \end{pmatrix} dy \\
&= \int_{-2}^{4} \left(y+1\right) - \left(\frac{1}{2}y^2 - 3\right) dy \\
&= \int_{-2}^{4} -\frac{1}{2}y^2 + y + 4\, dy \\
&= \left(-\frac{1}{6}y^3 + \frac{1}{2}y^2 + 4y\right)\Bigg|_{-2}^{4} \\
&= 18
\end{aligned}
$$

This is the same that we got using the first formula and this was definitely easier than the first method.

So, in this last example we've seen a case where we could use either formula to find the area. However, the second was definitely easier.

Students often come into a calculus class with the idea that the only easy way to work with functions is to use them in the form *y=f(x)*. However, as we've seen in this previous example there are definitely times when it will be easier to work with functions in the form *x=f(y)*. Make sure that you can deal with them in this form.

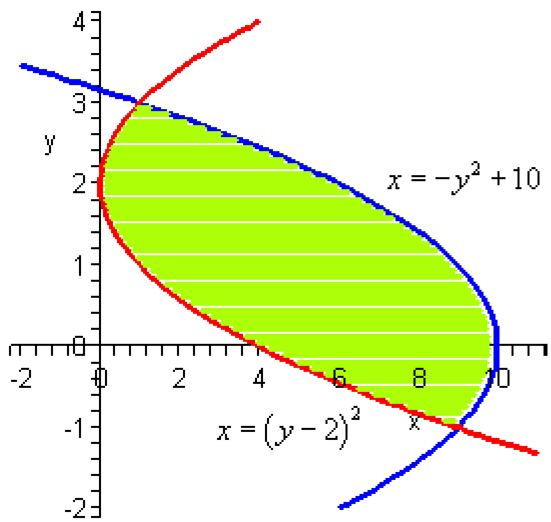
Let's take a look at one more example of this.

Example 7 Determine the area of the region bounded by $x = -y^2 + 10$ and $x = \left(y-2\right)^2$.

Solution
First, we will need intersection points.

$$
\begin{aligned}
-y^2 + 10 &= \left(y-2\right)^2 \\
-y^2 + 10 &= y^2 - 4y + 4 \\
0 &= 2y^2 - 4y - 6 \\
0 &= 2\left(y+1\right)\left(y-3\right)
\end{aligned}
$$

The intersection points are *y=-1* and *y=3*. Here is a sketch of the region.



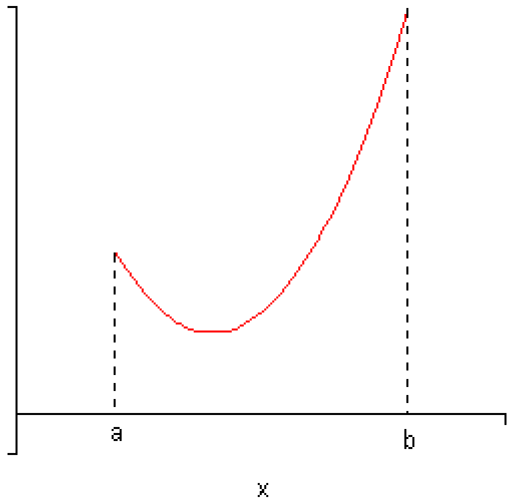
This is definitely a region where the second area formula will be easier. If we used the first there would be three different regions that we'd have to look at.

The area in this case is,

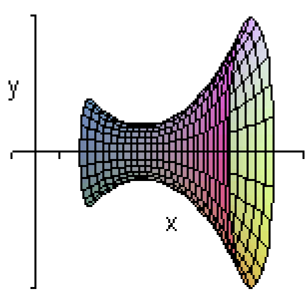
$$
\begin{aligned}
A &= \int_c^d \begin{pmatrix}\text{right} \\ \text{function}\end{pmatrix} - \begin{pmatrix}\text{left} \\ \text{function}\end{pmatrix} dy \\
&= \int_{-1}^{3} -y^2 + 10 - (y-2)^2 \, dy \\
&= \int_{-1}^{3} -2y^2 + 4y + 6 \, dy \\
&= \left. \left(-\frac{2}{3} y^3 + 2y^2 + 6y \right) \right|_{-1}^{3} \\
&= \frac{64}{3}
\end{aligned}
$$

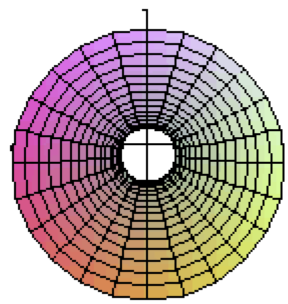
Volumes of Solids of Revolution / Method of Rings

In this section we will start looking at the volume of a solid of revolution. We should first define just what a solid of revolution is. To get a solid of revolution we start out with a function, *y=f(x)*, on an interval [*a*,*b*].



We then rotate this curve about a given axis to get the surface of the solid of revolution. For purposes of this discussion let's rotate the curve about the *x*-axis, although it could be any vertical or horizontal axis. Doing this gives the following three dimensional region.





The image on the right shows the 3-D object from the side and the image on the left shows the 3-D object from the end. Note that we didn't cap the ends here and are really just showing the surface of the solid of revolution. We didn't cap the ends in an effort to (hopefully) make the solid a little clearer. This is especially true for the end view. Had we capped it all you would have seen is a circle, at least this way you can "see" (in some way) that we don't just have a circle.

We want to determine the volume of this object.

In the final section of the Extras chapter we derived the following formulas for the volume of this solid.

$$V = \int_a^b A(x)\,dx \qquad\qquad V = \int_c^d A(y)\,dy$$

where, $A(x)$ or $A(y)$ is the cross-sectional area of the solid. There are many ways to get the cross-sectional area and whether we will use $A(x)$ or $A(y)$ will depend upon the method and the axis of rotation used for each problem.

One of the easier methods for getting the cross-sectional area is to cut the object perpendicular to the axis of rotation. Doing this the cross section will be either a solid disk if the object is solid (as our above example is) or a ring if we've hollowed out a portion of the solid (we will see this eventually).

In the case that we get a solid disk the area is,

$$A = \pi\left(\text{radius}\right)^2$$

where the radius will depend upon the function and the axis of rotation.

In the case that we get a ring the area is,

$$A = \pi\left(\begin{pmatrix}\text{outer}\\ \text{radius}\end{pmatrix}^2 - \begin{pmatrix}\text{inner}\\ \text{radius}\end{pmatrix}^2\right)$$

where again both of the radius will depend on the functions given and the axis of rotation.

Also, in both cases whether the area is a function of x or a function of y will depend upon the axis of rotation as we will see.

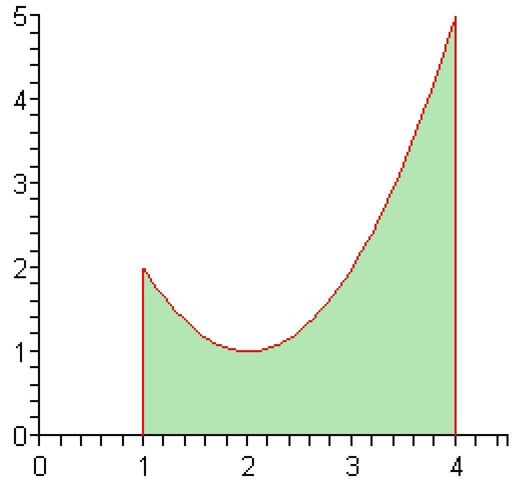
This method is often called the **method of disks** or the **method of rings**.

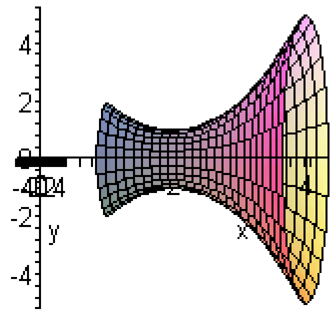
Let's do an example.

Example 1 Determine the volume of the solid obtained by rotating the region bounded by $y = x^2 - 4x + 5$, $x = 1$, $x = 4$, and the x-axis about the x-axis.

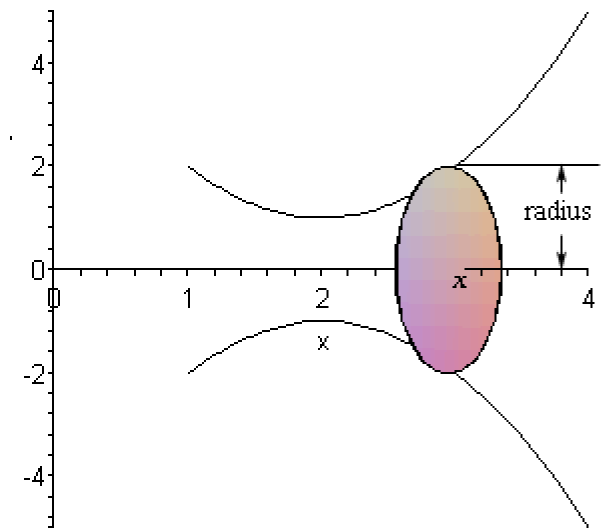
Solution

The first thing to do is get a sketch of the bounding region and the solid obtained by rotating the region about the x-axis.





Okay, to get a cross section we cut the object at any x. Below is a sketch of the upper and lower boundary of the object as well as a typical disk that we will get.



In this case the radius is simply the distance from the x-axis to the curve and this is nothing more than the function value at that particular x. The cross-sectional area is then,

$$A(x) = \pi\left(x^2 - 4x + 5\right)^2 = \pi\left(x^4 - 8x^3 + 26x^2 - 40x + 25\right)$$

Next we need to determine the limits of integration. Working from left to right the first cross section will occur at $x=1$ and the last cross section will occur at $x=4$. These are the limits of integration.

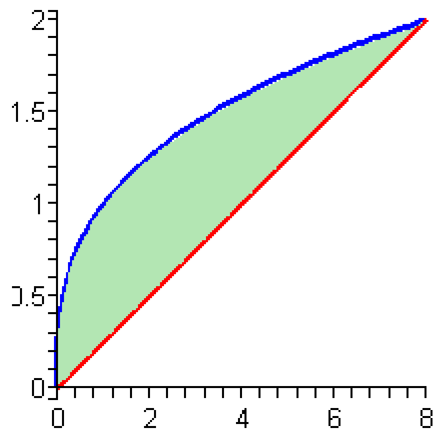
The volume of this solid is then,

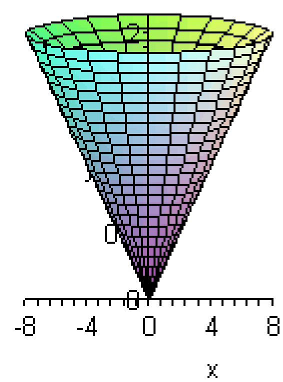
$$
\begin{aligned}
V &= \int_a^b A(x)\,dx \\
&= \pi\int_1^4 x^4 - 8x^3 + 26x^2 - 40x + 25\,dx \\
&= \pi\left(\frac{1}{5}x^5 - 2x^4 + \frac{26}{3}x^3 - 20x^2 + 25x\right)\Bigg|_1^4 \\
&= \frac{78\pi}{5}
\end{aligned}
$$

Example 2 Determine the volume of the solid obtained by rotating the portion of the region bounded by $y = \sqrt[3]{x}$ and $y = \dfrac{x}{4}$ that lies in the first quadrant about the *y*-axis.

Solution

First, let's get a graph of the bounding region and a graph of the object. Remember that we only want the portion of the bounding region that lies in the first quadrant. There is a portion of the bounding region that is in the third quadrant as well, but we don't want that for this problem.



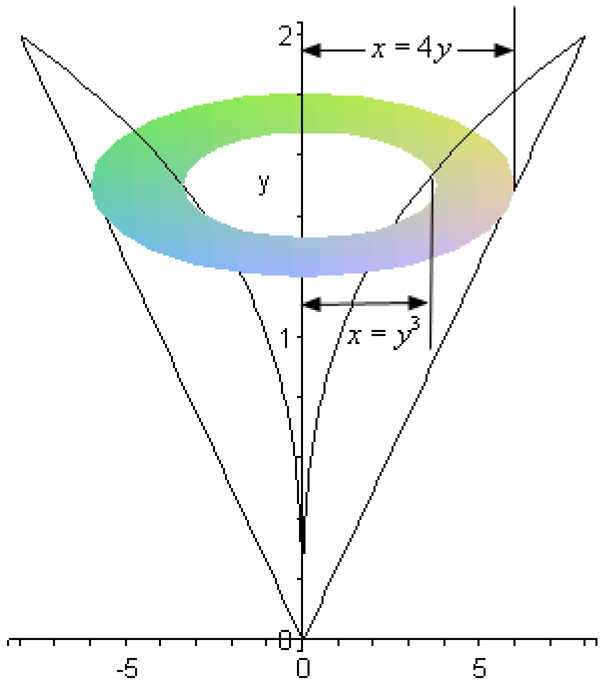


There are a couple of things to note with this problem. First, we are only looking for the volume of the "walls" of this solid, not the complete interior as we did in the last example.

Next, we will get our cross section by cutting the object perpendicular to the axis of rotation. This means that the cross section will be a ring (remember we are only looking at the walls) and it will be horizontal at some *y*. This means that the inner and outer radius for the ring will be *x* values and so we will need to rewrite our functions into the form *x=f(y)*. Here are the functions written in the correct form,

$$
\begin{aligned}
y &= \sqrt[3]{x} & &\Rightarrow & x &= y^3 \\
y &= \frac{x}{4} & &\Rightarrow & x &= 4y
\end{aligned}
$$

Here is a sketch of the boundaries of the walls of this object as well as a typical ring.



The cross-sectional area is then,

$$A(y) = \pi\left((4y)^2 - (y^3)^2\right) = \pi\left(16y^2 - y^6\right)$$

The first cross-section will occur at $y=0$ and the last cross-section will occur at $y=2$. These will be the limits of integration.

The volume is,

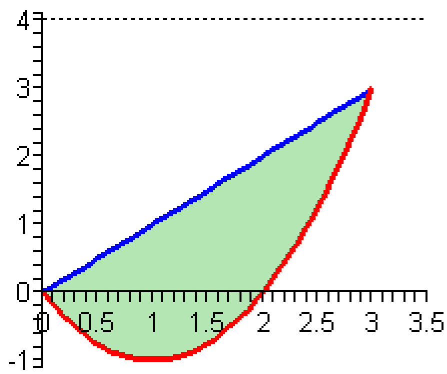
$$\begin{aligned} V &= \int_c^d A(y)\,dy \\ &= \pi\int_0^2 16y^2 - y^6\,dy \\ &= \pi\left(\frac{16}{3}y^3 - \frac{1}{7}y^7\right)\Bigg|_0^2 \\ &= \frac{512\pi}{21} \end{aligned}$$

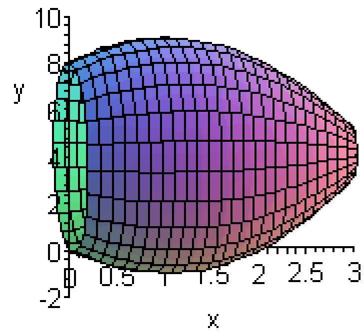
With these two examples out of the way we can now make a generalization about this method. If we rotate about a horizontal axis (the *x*-axis for example) then the cross-sectional area will be a function of *x*. Likewise, if we rotate about a vertical axis (the *y*-axis for example) then the cross-sectional area will be a function of *y*.

The remaining two examples in this section will make sure that we don't get too used to the idea of always rotating about the x or y-axis. There are other horizontal or vertical axis's that we can rotate about.

Example 3 Determine the volume of the solid obtained by rotating the region bounded by $y = x^2 - 2x$ and $y = x$ about the line $y = 4$.

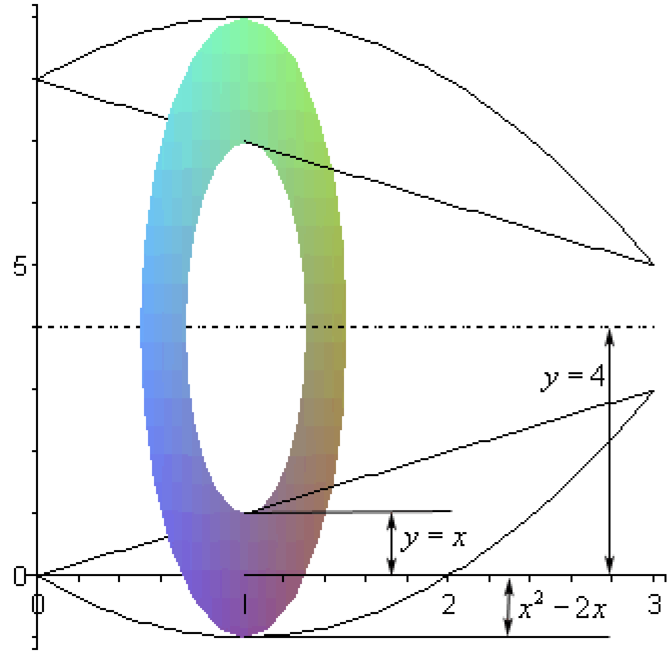
Solution
First let's get the bounding region and the solid graphed.





Again, we are going to be looking for the volume of the walls of this object. Also since we are rotating about a horizontal axis we know that the cross-sectional area will be a function of x.

Here is a sketch of the boundaries of the walls as well as a typical ring for this solid,



Now, we're going to have to be careful here in determining the inner and outer radius.

Let's start with the inner radius. The inner radius is NOT x. The distance from the x-axis to the inner edge of the ring is x, but we want the radius and that is the distance from the axis of rotation to the inner edge of the ring. The inner radius is then,

$$\text{inner radius} = 4 - x$$

Likewise, the outer radius is,

$$\text{outer radius} = 4 - \left(x^2 - 2x\right) = -x^2 + 2x + 4$$

Note that this also works for the ring that is shown where the outer edge is actually below the x-axis. At this point the value of the lower function is in fact negative and so four minus a negative number will end up being an addition and we will get the correct radius.

The cross-sectional area is then,

$$A(x) = \pi\left(\left(-x^2 + 2x + 4\right)^2 - \left(4 - x\right)^2\right) = \pi\left(x^4 - 4x^3 - 5x^2 + 24x\right)$$

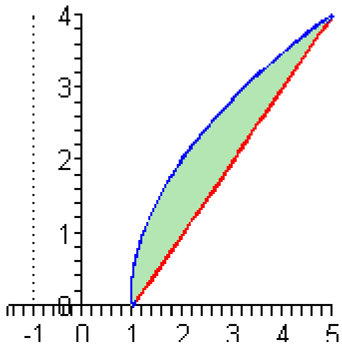
The first ring will occur at $x=0$ and the last ring will occur at $x=3$ and so these are our limits of integration.

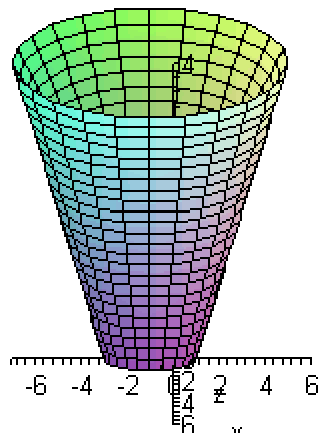
The volume is,

$$V = \int_a^b A(x)\,dx$$
$$= \pi \int_0^3 x^4 - 4x^3 - 5x^2 + 24x\,dx$$
$$= \pi \left(\frac{1}{5}x^5 - x^4 - \frac{5}{3}x^3 + 12x^2 \right) \Bigg|_0^3$$
$$= \frac{153\pi}{5}$$

Example 4 Determine the volume of the solid obtained by rotating the region bounded by $y = 2\sqrt{x-1}$ and $y = x-1$ about the line $x = -1$.

Solution
As with the previous examples, let's first graph the bounded region and the solid.

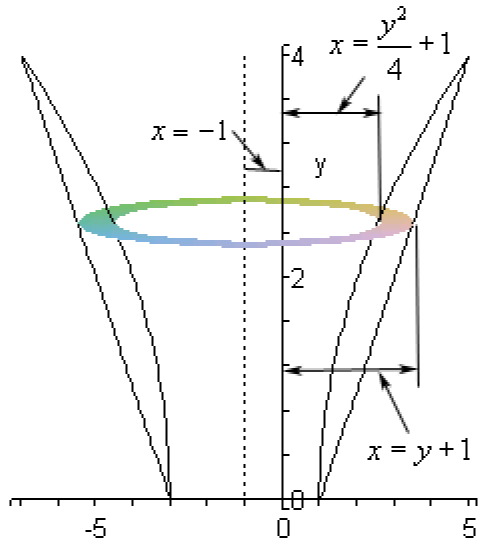




Now, let's notice that since we are rotating about a vertical axis and so the cross-sectional area will be a function of *y*. This also means that we are going to have to rewrite the functions to also get them in terms of *y*.

$$y = 2\sqrt{x-1} \qquad \Rightarrow \qquad x = \frac{y^2}{4} + 1$$
$$y = x - 1 \qquad \Rightarrow \qquad x = y + 1$$

Now, let's sketch a typical ring.



In this case it looks we've got the following for the inner and outer radius.

$$\text{outer radius} = y + 1 + 1 = y + 2$$
$$\text{inner radius} = \frac{y^2}{4} + 1 + 1 = \frac{y^2}{4} + 2$$

Note that we had to add one on to both of them to take into account the fact that axis of rotation was a distance of one from the *y*-axis.

The cross-sectional area it then,

$$A(y) = \pi\left((y+2)^2 - \left(\frac{y^2}{4} + 2\right)^2\right) = \pi\left(4y - \frac{y^4}{16}\right)$$

The first ring will occur at *y=0* and the final ring will occur at *y=4* and so these will be our limits of integration.

The volume is,

$$\begin{aligned} V &= \int_c^d A(y)\,dy \\ &= \pi\int_0^4 4y - \frac{y^4}{16}\,dy \\ &= \pi\left(2y^2 - \frac{1}{80}y^5\right)\Big|_0^2 \\ &= \frac{96\pi}{5} \end{aligned}$$

Volumes of Solids of Revolution / Method of Cylinders

In the previous section we started looking at finding volumes of solids of revolution. In that section we took cross sections that were rings or disks, found the cross-sectional area and then used the following formulas to find the volume of the solid.

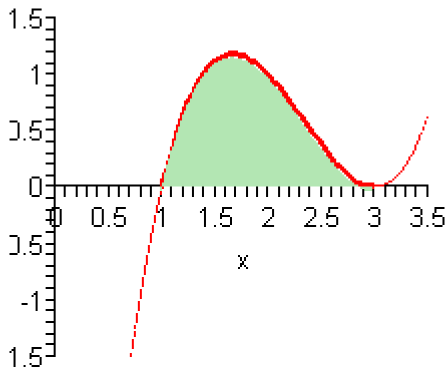
$$V = \int_a^b A(x)\,dx \qquad\qquad V = \int_c^d A(y)\,dy$$

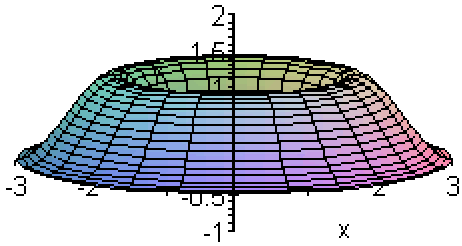
However, using rings/disks will not always be the easiest method to determine the volume. Consider the following example.

Example 1 Determine the volume of the solid obtained by rotating the region bounded by $y = (x-1)(x-3)^2$ and the *x*-axis about the *y*-axis.

Solution

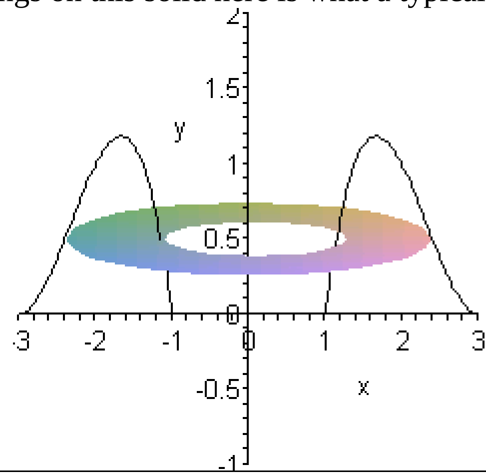
As we did in the previous section, let's first graph the bounded region and solid.





So, we've basically got something that's roughly doughnut shaped.

Now, if we were to use rings on this solid here is what a typical ring would look like.



There are several problems with this.

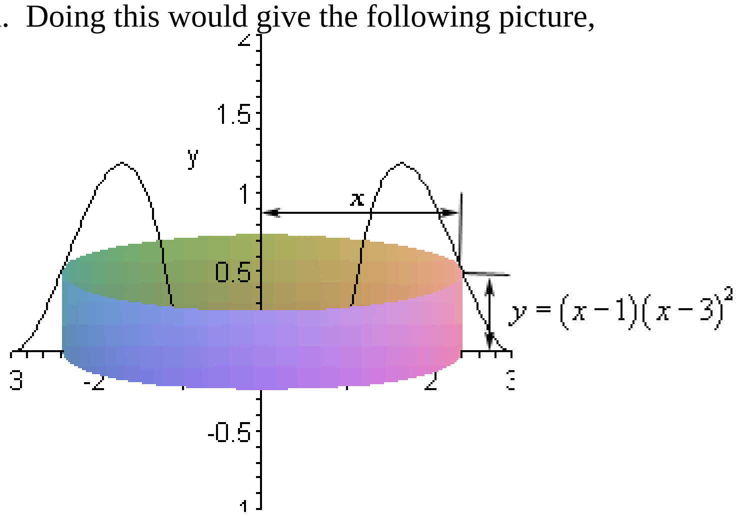
First, both the inner and outer radius are defined by the same function. This in itself can be dealt with on occasion. In fact, we did something like this back in the Area Between Curves section. However, this usually means more work than other methods and so it's often not the best approach.

This leads to the second problem. In order to use rings we would need to put this function into the form $x=f(y)$. That is NOT easy in general for a cubic polynomial so that alone will prevent us from using rings.

The last problem is not so much a problem as its just added work. If we were to use rings the limit would be y limits and this means that we will need to know how high the graph goes. To this point the limits of integration have always been intersection points that were fairly easy to find. However, in this case the highest point is not an intersection point. We do know how to find this point. We spent several sections in the Applications of Derivatives chapter talking about how to find maximum values of functions. However, that can be a fair about of work and so it would be best to avoid it if possible.

So, we've seen three problems that will either increase our work load or outright prevent us from using rings in this case.

What we need to do is to find a different way to cut the solid that will give us a cross-sectional area that we can work with. One way to do this is to think of our solid as a lump of cookie dough and instead of cutting it perpendicular to the axis of rotation we could instead center a cylindrical cookie cutter on the axis of rotation and push this down into the solid. Doing this would give the following picture,



Doing this gives us a cylinder or shell in the object and we can easily find its surface area. We can use this in volume formulas. The surface area of this cylinder is,

$$\begin{aligned} A(x) &= 2\pi(\text{radius})(\text{height}) \\ &= 2\pi(x)\left((x-1)(x-3)^2\right) \\ &= 2\pi\left(x^4 - 7x^3 + 15x^2 - 9x\right) \end{aligned}$$

Now we need limits of integration. The key here is to recognize that as we move from the lower limit of integration to the upper limit of integration we need to cover the complete solid with cylinders.

So, the first cylinder will cut into the solid at *x=1* and as we increase *x* to *x=3* we will completely cover both sides of the solid since expanding the cylinder in one direction will automatically expand it in the other direction as well.

The volume of this solid is then,

$$\begin{aligned} V &= \int_a^b A(x)\,dx \\ &= 2\pi\int_1^3 x^4 - 7x^3 + 15x^2 - 9x\,dx \\ &= 2\pi\left(\frac{1}{5}x^5 - \frac{7}{4}x^4 + 5x^3 - \frac{9}{2}x^2\right)\Bigg|_1^3 \\ &= \frac{24\pi}{5} \end{aligned}$$

The method used in the last example is called the **method of cylinders** or **method of shells**. The formula for the area in all cases will be,

$$A = 2\pi(\text{radius})(\text{height})$$

There are a couple of important differences between this method and the method of rings/disks.

First, rotation about a vertical axis will give an area that is a function of *x* and rotation about a horizontal axis will give an area that is a function of *y*. This is exactly opposite of the method of rings/disks.

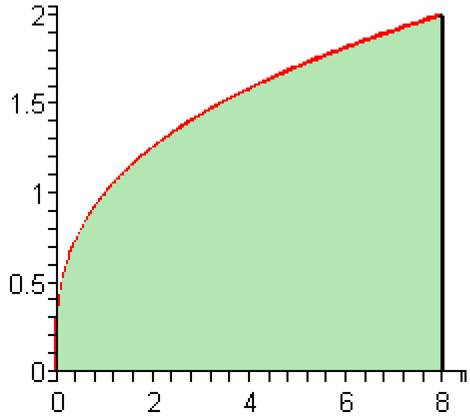
Second, we don't take the complete range of *x* or *y* for the limits of integration as we did in the previous section. Instead we take a range of *x* or *y* that will cover one side of the solid.

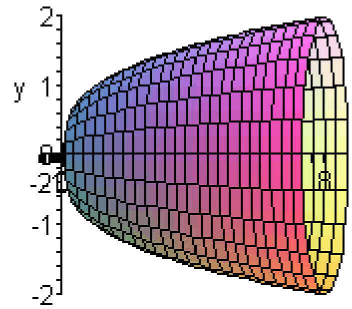
Let's take a look at some another example.

Example 2 Determine the volume of the solid obtained by rotating the region bounded by $y = \sqrt[3]{x}$, $x = 8$ and the *x*-axis about the *x*-axis.

Solution

First let's get a graph of the bounded region and the solid.

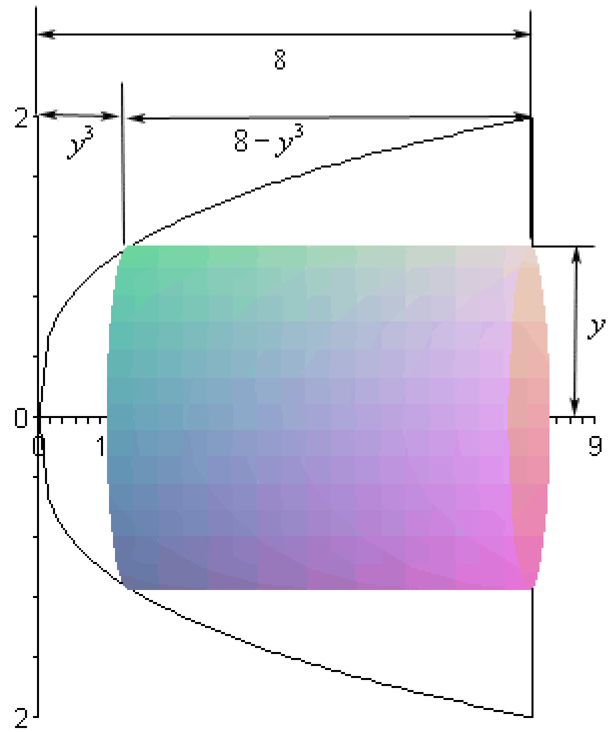




Okay, we are rotating about a horizontal axis. This means that the area will be a function of *y* and so our equation will also need to be wrote in *x=f(y)* form.

$$x = \sqrt[3]{x} \qquad \Rightarrow \qquad x = y^3$$

Here is a sketch of a typical cylinder.



The area here is

$$
\begin{aligned}
A(y) &= 2\pi(\text{radius})(\text{height}) \\
&= 2\pi(y)\left(8-y^3\right) \\
&= 2\pi\left(8y-y^4\right)
\end{aligned}
$$

The first cylinder will cut into the solid at *y=0* and the final cylinder will cut in at *y=2* and so these are our limits.

The volume is,

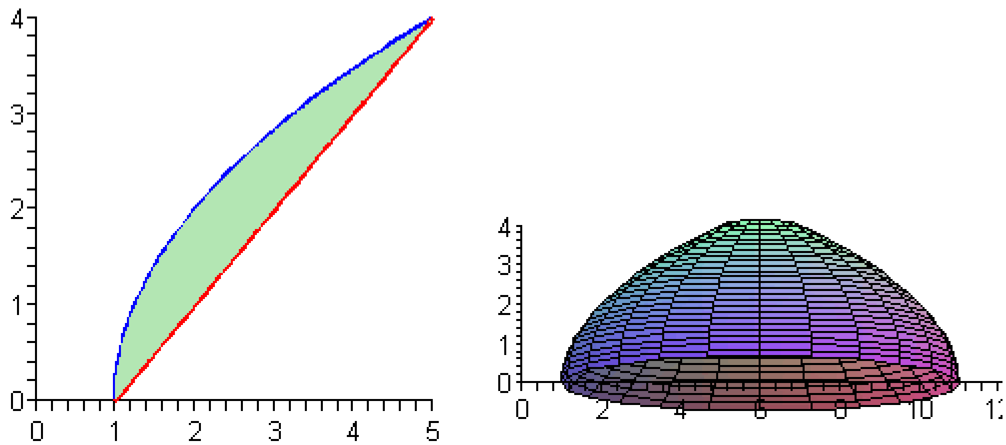
$$
\begin{aligned}
V &= \int_c^d A(y)\,dy \\
&= 2\pi\int_0^2 8y-y^4\,dy \\
&= 2\pi\left(4y^2-\frac{1}{5}y^5\right)\Bigg|_0^2 \\
&= \frac{96\pi}{5}
\end{aligned}
$$

The remaining examples in this section will have axis of rotation about axis other than the *x* and *y*-axis. As with the method of rings/disks we will need to be a little careful with these.

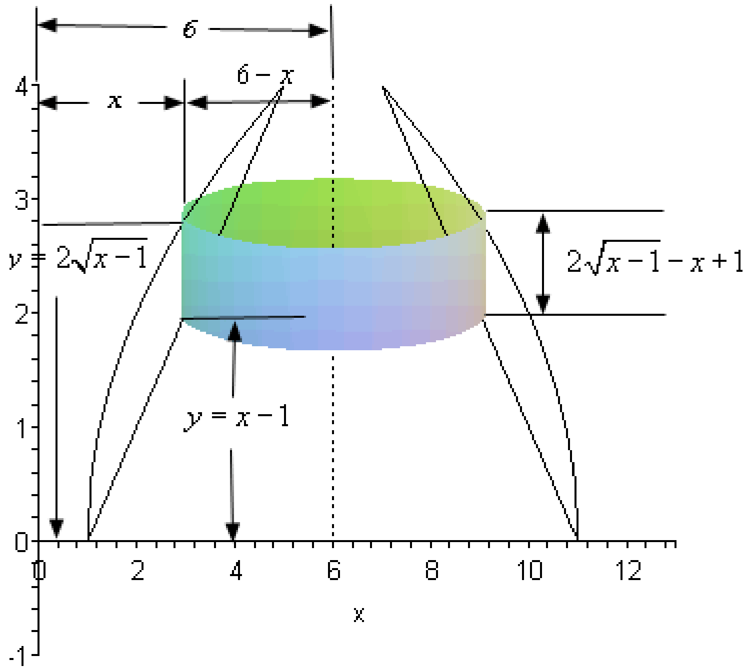
Example 3 Determine the volume of the solid obtained by rotating the region bounded by $y = 2\sqrt{x-1}$ and $y = x-1$ about the line $x = 6$.

Solution

Here's a graph of the bounded region and solid.



Here is a sketch of a typical cylinder.



Okay, there is a lot going on in this sketch. First notice that the radius is not just an *x* or *y* as it was in the previous two cases. In this case *x* is the distance from the *x*-axis to the edge of the cylinder and we need the distance from the axis of rotation to the edge of the cylinder. That means that the radius of this cylinder is *6-x*.

Next, the height of the cylinder is the difference of the two functions.

The area is then,

$$\begin{aligned} A(x) &= 2\pi(\text{radius})(\text{height}) \\ &= 2\pi(6-x)\left(2\sqrt{x-1}-x+1\right) \\ &= 2\pi\left(x^2-7x+6+12\sqrt{x-1}-2x\sqrt{x-1}\right) \end{aligned}$$

Now the first cylinder will cut into the solid at *x=1* and the final cylinder will cut into the solid at *x=5* so there are our limits.

Here is the volume.

$$\begin{aligned} V &= \int_a^b A(x)\,dx \\ &= 2\pi\int_1^5 x^2 - 7x + 6 + 12\sqrt{x-1} - 2x\sqrt{x-1}\,dx \\ &= 2\pi\left(\frac{1}{3}x^3 - \frac{7}{2}x^2 + 6x + 8(x-1)^{\frac{3}{2}} - \frac{4}{3}(x-1)^{\frac{3}{2}} - \frac{4}{5}(x-1)^{\frac{5}{2}}\right)\Bigg|_1^5 \\ &= 2\pi\left(\frac{136}{15}\right) \\ &= \frac{272\pi}{15} \end{aligned}$$

The integration of the last term is a little tricky so let's do that here. It will use the substitution,

$$u = x - 1 \qquad du = dx \qquad x = u + 1$$

$$\begin{aligned} \int 2x\sqrt{x-1}\,dx &= 2\int (u+1)u^{\frac{1}{2}}\,du \\ &= 2\int u^{\frac{3}{2}} + u^{\frac{1}{2}}\,du \\ &= 2\left(\frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}}\right) + c \\ &= \frac{4}{5}(x-1)^{\frac{5}{2}} + \frac{4}{3}(x-1)^{\frac{3}{2}} + c \end{aligned}$$

We saw one of these kinds of substitutions back in the substitution section.

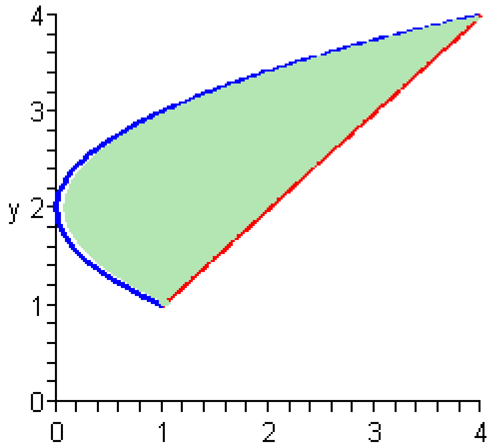
Example 4 Determine the volume of the solid obtained by rotating the region bounded by $x = (y-2)^2$ and $y = x$ about the line $y = -1$.

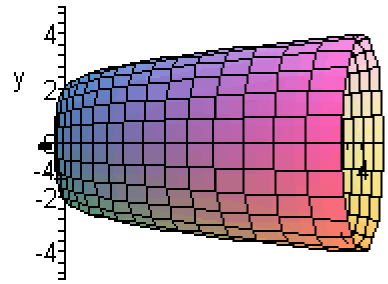
Solution

We should first get the intersection points there.

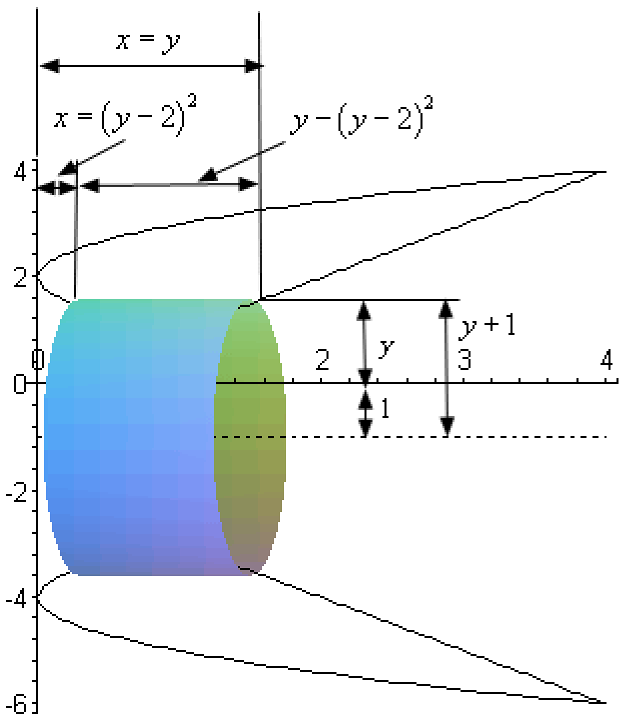
$$\begin{aligned} y &= (y-2)^2 \\ y &= y^2 - 4y + 4 \\ 0 &= y^2 - 5y + 4 \\ 0 &= (y-4)(y-1) \end{aligned}$$

So, the two curves will intersect at *y=1* and *y=4*. Here is a sketch of the bounded region and the solid.





Here is a typical cylinder for this solid.



Here's the area for this cylinder.

$$\begin{aligned} A(y) &= 2\pi(\text{radius})(\text{height}) \\ &= 2\pi(y+1)\left(y-(y-2)^2\right) \\ &= 2\pi\left(-y^3+4y^2+y-4\right) \end{aligned}$$

The first cylinder will cut into the solid at *y=1* and the final cylinder will cut in at *y=4*. The volume is then,

$$\begin{aligned} V &= \int_c^d A(y)\,dy \\ &= 2\pi\int_1^4 -y^3 + 4y^2 + y - 4\,dy \\ &= 2\pi\left(-\frac{1}{4}y^4 + \frac{4}{3}y^3 + \frac{1}{2}y^2 - 4y\right)\Big|_1^4 \\ &= \frac{63\pi}{2} \end{aligned}$$

Work

This is the final application of integral that we'll be looking at in this course. In this section we will be looking at the amount of work that is done by a force in moving an object.

In a first course in Physics you typically look at the work that a constant force, F, does when moving an object over a distance of d. In these cases the work is,

$$W = Fd$$

However, most forces are not constant and will depend upon where exactly the force is acting. So, let's suppose that the force at any x is given by $F(x)$. Then the work done by the force in moving an object from $x=a$ to $x=b$ is given by,

$$W = \int_a^b F(x)\,dx$$

Notice that if the force constant we get the correct formula for a constant force.

$$\begin{aligned} W &= \int_a^b F\,dx \\ &= Fx\Big|_a^b \\ &= F(b-a) \end{aligned}$$

where b-a is simply the distance moved, or d.

So, let's take a look at a couple of examples of non-constant forces.

Example 1 A spring has a natural length of 20 cm. A 40 N force is required to stretch (and hold the spring) to a length of 30 cm. How much work is done in stretching the spring from 35 cm to 38 cm?

Solution

This example will require Hooke's Law to determine the force. Hooke's Law tells us that the force required to stretch a spring a distance of x meters from its natural length is,

$$F(x) = k\,x$$

where *k>0* is called the spring constant.

The first thing that we need to do is determine the spring constant for this spring. We can do that using the initial information. A force of 40 N is required to stretch the spring 30cm-20cm = 10cm = 0.10m from its natural length. Using Hooke's Law we have,

$$40 = 0.10k \qquad \Rightarrow \qquad k = 400$$

So, according to Hooke's Law the force required to hold this spring *x* meters from its natural length is,

$$F(x) = 400x$$

We want to know the work required to stretch the spring from 35cm to 38cm. First we need to convert these into distances from the natural length in meters. Doing that gives us *x*'s of 0.15m and 0.18m.

The work is then,

$$\begin{aligned} W &= \int_{0.15}^{0.18} 400x\,dx \\ &= 200x^2 \Big|_{0.15}^{0.18} \\ &= 1.98\,\mathrm{J} \end{aligned}$$

Example 2 We have a cable that weighs 2 lbs/ft attached to a bucket filled with coal that weighs 800 lbs. The bucket is initially at the bottom of a 500 ft mine shaft. Answer each of the following about this.

- **(a)** Determine the amount of work required to lift the bucket to the mid point of the shaft.
- **(b)** Determine the amount of work required to lift the bucket from the mid point of the shaft to the top of the shaft.
- **(c)** Determine the amount of work required to lift the bucket all the way up the shaft.

Solution

Before answering either part we first need to determine the force. In this case the force will be the weight of the bucket and cable at any point in the shaft.

To determine a formula for this we will first need to set a convention for *x*. For this problem we will set *x* to be the amount of cable that has been pulled up. So at the bottom of the shaft *x=0*, at the mid point of the shaft *x=250* and at the top of the shaft *x=500*. Also at any point in the shaft there is *500-x* feet of cable still in the shaft.

The force then for any *x* is,

$$\begin{aligned} F(x) &= \text{weight of cable} + \text{ weight of bucket/coal} \\ &= 2(500 - x) + 800 \\ &= 1800 - 2x \end{aligned}$$

We can now answer the questions.
(a) In this case we want to know the work required to move the cable and bucket/coal from $x=0$ to $x=250$. The work required is,

$$\begin{aligned} W &= \int_0^{250} F(x)\,dx \\ &= \int_0^{250} 1800 - 2x\,dx \\ &= \left(1800x - x^2\right)\Big|_0^{250} \\ &= 387500 \end{aligned}$$

(b) In this case we want to move the cable and bucket/coal from $x=250$ to $x=500$. The work required is,

$$\begin{aligned} W &= \int_{250}^{500} F(x)\,dx \\ &= \int_{250}^{500} 1800 - 2x\,dx \\ &= \left(1800x - x^2\right)\Big|_{250}^{500} \\ &= 262500 \end{aligned}$$

(c) In this case the work is,

$$\begin{aligned} W &= \int_0^{500} F(x)\,dx \\ &= \int_0^{500} 1800 - 2x\,dx \\ &= \left(1800x - x^2\right)\Big|_0^{500} \\ &= 650000 \end{aligned}$$

Note that we could have instead just added the results from the first two parts and we would have gotten the same answer.

Extras

Introduction

Because this material is also being prepared for presentation on the web there were several topics that, if presented in the appropriate sections, would have made those sections too large for web presentation.

This chapter contains those topics.

Proof of Product Rule – The proof of the product rule for derivatives.

Proof of Quotient Rule – The proof of the quotient rule for derivatives.

Types of Infinity – This is a discussion on the types of infinity and how these affect certain limits.

Summation Notation – Here is a quick review of summation notation.

Constant of Integration – This is a discussion on a couple of subtleties involving constants of integration that many students don't think about.

Area and Volume Formulas – Here is the derivation of the formulas for finding area between two curves and finding the volume of a solid of revolution.

Proof of Product Rule

Okay, let's start with our two functions *f(x)* and *g(x)*. The only way to prove the product rule is to use the definition of the derivative so we'll need to do that.

$$(f\,g)' = \lim_{h\to 0} \frac{f(x+h)\,g(x+h) - f(x)\,g(x)}{h}$$

Now, on the surface this does nothing for us. We'll need to manipulate things a little. What we'll do is subtract out and add in $f(x+h)\,g(x)$ to the numerator. Note that we're really just adding in a zero here since these two terms will cancel. This will give us,

$$(f\,g)' = \lim_{h\to 0} \frac{f(x+h)\,g(x+h) - f(x+h)\,g(x) + f(x+h)\,g(x) - f(x)\,g(x)}{h}$$

Notice that we added the two terms into the middle of the numerator. As written we can break up the limit into two pieces. From the first piece we can factor a $f(x+h)$ out and we can factor a $g(x)$ out of the second piece. Doing this gives,

$$\begin{aligned}(f\,g)' &= \lim_{h\to 0} \frac{f(x+h)\big(g(x+h) - g(x)\big)}{h} + \lim_{h\to 0} \frac{g(x)\big(f(x+h) - f(x)\big)}{h} \\ &= \lim_{h\to 0} f(x+h)\frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} g(x)\frac{f(x+h) - f(x)}{h}\end{aligned}$$

At this point both of the limits are a limit of a product and we can do that as a product of limits.

$$\left(f\,g\right)' = \left(\lim_{h\to 0} f\left(x+h\right)\right)\left(\lim_{h\to 0}\frac{g\left(x+h\right)-g\left(x\right)}{h}\right)+\left(\lim_{h\to 0} g\left(x\right)\right)\left(\lim_{h\to 0}\frac{f\left(x+h\right)-f\left(x\right)}{h}\right)$$

The individual limits in this are,

$$\begin{aligned}\lim_{h\to 0}\frac{g\left(x+h\right)-g\left(x\right)}{h} &= g'\left(x\right)\\ \lim_{h\to 0}\frac{f\left(x+h\right)-f\left(x\right)}{h} &= f'\left(x\right)\\ \lim_{h\to 0} f\left(x+h\right) &= f\left(x\right)\\ \lim_{h\to 0} g\left(x\right) &= g\left(x\right)\end{aligned}$$

The first two limits are nothing more than the definition the derivative for *g(x)* and *f(x)* respectively. The third limit we get simply by plugging in *h=0*. The forth limit seems a little tricky. Recall that the limit of a constant is just the constant. Well since the limit is only concerned with allowing *h* to go to zero as far as its concerned *g(x)* is a constant. Note that the function is probably not a constant, but because it contains no *h*'s will be a constant as far as the limit is concerned.

Plugging all these into the last step gives us,

$$\left(f\,g\right)' = f\left(x\right)g'\left(x\right)+g\left(x\right)f'\left(x\right)$$

And there is the product rule. The terms are written in the opposite order than what we originally gave the product rule as, but they can always be rearranged to look like the rule as we gave it.

Proof of Quotient Rule

Okay, let's start with our two functions *f(x)* and *g(x)*. We'll need to plug into the definition of the derivative to prove the quotient rule so let's do that.

$$\begin{aligned}\left(\frac{f}{g}\right)' &= \lim_{h\to 0}\frac{\dfrac{f\left(x+h\right)}{g\left(x+h\right)}-\dfrac{f\left(x\right)}{g\left(x\right)}}{h}\\ &= \lim_{h\to 0}\frac{1}{h}\,\frac{f\left(x+h\right)g\left(x\right)-f\left(x\right)g\left(x+h\right)}{g\left(x+h\right)g\left(x\right)}\end{aligned}$$

To make our life a little easier we moved the h in the denominator of the first step out to the front as a $\frac{1}{h}$. We also wrote the numerator as a single rational expression. This step is required to make this proof work.

Now, for the next step will need to subtract out and add in $f(x)g(x)$ to the numerator.

$$\left(\frac{f}{g}\right)' = \lim_{h\to 0}\frac{1}{h}\,\frac{f(x+h)g(x)-f(x)g(x)+f(x)g(x)-f(x)g(x+h)}{g(x+h)g(x)}$$

The next step is to rewrite the two fractions as follows,

$$\left(\frac{f}{g}\right)' = \lim_{h\to 0}\frac{1}{g(x+h)g(x)}\,\frac{f(x+h)g(x)-f(x)g(x)+f(x)g(x)-f(x)g(x+h)}{h}$$

All we did was interchange the two denominators. Since we are multiplying the fractions we can do this.

Now, the larger fraction can be broken up as follows.

$$\left(\frac{f}{g}\right)' = \lim_{h\to 0}\frac{1}{g(x+h)g(x)}\left(\frac{f(x+h)g(x)-f(x)g(x)}{h}+\frac{f(x)g(x)-f(x)g(x+h)}{h}\right)$$

In the first fraction we will factor a $g(x)$ out and in the second we will factor a $-f(x)$ out. This gives,

$$\left(\frac{f}{g}\right)' = \lim_{h\to 0}\frac{1}{g(x+h)g(x)}\left(g(x)\frac{f(x+h)-f(x)}{h}-f(x)\frac{g(x+h)-g(x)}{h}\right)$$

We can now take some limits.

$$\begin{aligned}\left(\frac{f}{g}\right)' = \frac{1}{\lim\limits_{h\to 0}g(x+h)\lim\limits_{h\to 0}g(x)}\Bigg(&\left(\lim_{h\to 0}g(x)\right)\left(\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right)- \\ &\left(\lim_{h\to 0}f(x)\right)\left(\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right)\Bigg)\end{aligned}$$

The individual limits are,

$$\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}=g'(x)$$
$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=f'(x)$$
$$\lim_{h\to 0}g(x+h)=g(x)$$
$$\lim_{h\to 0}g(x)=g(x)$$
$$\lim_{h\to 0}f(x)=f(x)$$

The first two limits are nothing more than the definition the derivative for *g(x)* and *f(x)* respectively. The third limit we get simply by plugging in *h=0*. The forth limit seems a little tricky. Recall that the limit of a constant is just the constant. Well since the limit is only concerned with allowing *h* to go to zero as far as its concerned *g(x)* is a constant. Note that the function is probably not a constant, but because it contains no *h*'s will be a constant as far as the limit is concerned. The fifth limit is found in the same way. Since there are no *h*'s in *f(x)* the function is a constant as far as the limit is concerned.

Plugging in the limits gives,

$$\begin{aligned}\left(\frac{f}{g}\right)' &= \frac{1}{g(x)g(x)}\left(g(x)f'(x)-f(x)g'(x)\right)\\ &= \frac{f'g-fg'}{g^2}\end{aligned}$$

And we have the quotient rule.

Types of Infinity

Most students have run across infinity at some point in time prior to a calculus class. However, when they have dealt with it, it was just a symbol used to represent a really, really large positive or really, really large negative number and that was the extent of it. Once they get into a calculus class students are asked to do some basic algebra with infinity and this is where they get into trouble. Infinity is NOT a number and for the most part doesn't behave like a number. However, despite that we'll think of infinity in this section as a really, really, really large number that is so large there isn't another number larger than it. This is not correct of course, but may help with the discussion in this section.

So, let's start thinking about addition with infinity. When you add two non-zero numbers you get a new number. For example, $4+7=11$. With infinity this is not true. With infinity you have the following.

$$\infty + a = \infty \qquad \text{where } a \neq -\infty$$
$$\infty + \infty = \infty$$

In other words, a really, really large positive number (∞) plus any positive number, regardless of the size, is still a really, really large positive number. Likewise, you can add a negative number (*i.e.* $a < 0$) to a really, really large positive number and stay really, really large and positive. So, addition involving infinity can be dealt with in an intuitive way if you're careful. Note as well that the *a* must NOT be negative infinity. If it is, there are some serious issues that we need to deal with.

Subtraction with negative infinity can also be dealt with in an intuitive way in most cases. A really, really large negative number minus any positive number, regardless of its size, is still a really, really large negative number. Subtracting a negative number (*i.e.* $a < 0$) from a really, really large negative number will still be a really, really large negative number. Or,

$$-\infty - a = -\infty \qquad \text{where } a \neq -\infty$$
$$-\infty - \infty = -\infty$$

Again, *a* must not be negative infinity to avoid some potentially serious difficulties.

Multiplication can be dealt with fairly intuitively as well. A really, really large number (positive, or negative) times any number, regardless of size, is still a really, really large number. In the case of multiplication we have

$$(a)(\infty) = \infty \qquad \text{if } a > 0$$
$$(a)(\infty) = -\infty \qquad \text{if } a < 0$$
$$(\infty)(\infty) = \infty$$
$$(-\infty)(-\infty) = \infty$$
$$(-\infty)(\infty) = -\infty$$

What you know about products of positive and negative numbers is still true here.

Some forms of division can be dealt with intuitively as well. A really, really large number divided by a number that isn't too large is still a really, really large number.

$$\frac{\infty}{a} = \infty \qquad \text{if } a > 0$$
$$\frac{\infty}{a} = -\infty \qquad \text{if } a < 0$$
$$\frac{-\infty}{a} = -\infty \qquad \text{if } a > 0$$
$$\frac{-\infty}{a} = \infty \qquad \text{if } a < 0$$

Division of a number by infinity is somewhat intuitive, but there are a couple of subtleties that you need to be aware of. When we talk about division by infinity we are really talking about a limiting process in which the denominator is going towards infinity. So, a number that isn't too large divided an increasingly large number is a increasingly small number. In other words in the limit we have,

$$\frac{a}{\infty} = 0$$

$$\frac{a}{-\infty} = 0$$

So, we've dealt with almost every basic algebraic operation involving infinity. There are two cases that that we haven't dealt with yet. These are

$$\infty - \infty = ?$$

$$\frac{\pm\infty}{\pm\infty} = ?$$

The problem with these two cases is that intuition doesn't really help here. A really, really large number minus a really, really large number can be anything ($-\infty$, a constant, or ∞). Likewise, a really, really large number divided by a really, really large number can also be anything ($\pm\infty$ - this depends on sign issues, 0, or a non-zero constant).

What we've got to remember here is that there are really, really large numbers and then there are really, really, really large numbers. In other words, some infinities are larger than other infinities. With addition, multiplication and the first sets of division we worked this wasn't an issue. The general size of the infinity just doesn't affect the answer in those cases. However, with the subtraction and division cases listed above, it does matter as we will see.

Here is one way to think of this idea that some infinities are larger than others. This is a fairly dry and technical way to think of this and your calculus problems will probably never use this stuff, but this it is a nice way of looking at this. Also, please note that I'm not trying to give a precise proof of anything here. I'm just trying to give you a little insight into the problems with infinity and how some infinities can be thought of as larger than others. For a much better (and definitely more precise) discussion see,

http://www.math.vanderbilt.edu/~schectex/courses/infinity.pdf

Let's start by looking at how many integers there are. Clearly, I hope, there are an infinite number of them, but let's try to get a better grasp on the "size" of this infinity. So, pick any two integers completely at random. Start at the smaller of the two and list, in increasing order, all the integers that come after that. Eventually we will reach the larger of the two integers that you picked.

Depending on the relative size of the two integers it might take a very, very long time to list all the integers between them and there isn't really a purpose to doing it. But, it could be done if we wanted to and that's the important part.

Because we could list all these integers between two randomly chosen integers we say that the integers are *countably infinite*. Again, there is no real reason to actually do this, it is simply something that can be done if we should chose to do so.

In general a set of numbers is called countably infinite if we can find a way to list them all out. In a more precise mathematical setting this is generally done with a special kind of function called a bijection that associates each number in the set with exactly one of the positive integers. To see some more details of this see the pdf give above.

It can also be shown that the set of all fractions are also countably infinite, although this is a little harder to show and is not really the purpose of this discussion. To see a proof of this see the pdf given above. It has a very nice proof of this fact.

Let's contrast this by trying to figure out how many numbers there are in the interval (0,1). By numbers, I mean all possible fractions that lie between zero and one as well as all possible decimals (that aren't fractions) that lie between zero and one. The following is similar to the proof given in the pdf above, but was nice enough and easy enough (I hope) that I wanted to include it here.

To start let's assume that all the numbers in the interval (0,1) are countably infinite. This means that there should be a way to list all of them out. We could have something like the following,

$$\begin{aligned} x_1 &= 0.692096\cdots \\ x_2 &= 0.171034\cdots \\ x_3 &= 0.993671\cdots \\ x_4 &= 0.045908\cdots \\ \vdots & \qquad \vdots \end{aligned}$$

Now, select the i^{th} decimal out of x_i as shown below

$$\begin{aligned} x_1 &= 0.\underline{6}92096\cdots \\ x_2 &= 0.1\underline{7}1034\cdots \\ x_3 &= 0.99\underline{3}671\cdots \\ x_4 &= 0.045\underline{9}08\cdots \\ \vdots & \qquad \vdots \end{aligned}$$

and form a new number with these digits. So, for our example we would have the number

$$x = 0.6739\cdots$$

In this new decimal replace all the 3's with a 1 and replace every other numbers with a 2. In the case of our example this would yield the new number

$$\bar{x} = 0.2212\cdots$$

Notice that this number is in the interval (0,1) and also notice that given how we choose the digits of the number this number will not be equal to the first number in our list, x_1, because the first digit of each is guaranteed to not be the same. Likewise, this new number will not get the same number as the second in our list, x_2, because the second digit of each is guaranteed to not be the same. Continuing in this manner we can see that this new number we constructed, $\bar{x}$, is guaranteed to not be in our listing. But this contradicts the initial assumption that we could list out all the numbers in the interval (0,1). Hence, it must not be possible to list out all the numbers in the interval (0,1).

Sets of numbers, such as all the numbers in (0,1), that we can't write down in a list are called *uncountably* infinite.

The reason for going over this is the following. An infinity that is uncountably infinite is significantly larger than an infinity that is only countably infinite. So, if we take the difference of two infinities we have a couple of possibilities.

$$\begin{aligned}
&\infty\left(\text{uncountable}\right)-\infty\left(\text{countable}\right)=\infty \\
&\infty\left(\text{countable}\right)-\infty\left(\text{uncountable}\right)=-\infty \\
&\infty\left(\text{countable}\right)-\infty\left(\text{countable}\right)=\text{ a constant}
\end{aligned}$$

Notice that we didn't put down a difference of two uncountable infinities. There is still have some ambiguity about just what the answer would be in this case, but that is a whole different topic.

We could also do something similar for quotients of infinities.

$$\begin{aligned}
&\frac{\infty\left(\text{countable}\right)}{\infty\left(\text{uncountable}\right)}=0 \\
&\frac{\infty\left(\text{uncountable}\right)}{\infty\left(\text{countable}\right)}=\infty \\
&\frac{\infty\left(\text{countable}\right)}{\infty\left(\text{countable}\right)}=\text{ a constant}
\end{aligned}$$

Again, we avoided a quotient of two uncountable infinities since there will still be ambiguities about its value.

Conclusion

Hopefully you've learned something from this discussion. Infinity simply isn't a number and because there are different kinds of infinity it generally doesn't behave as a number does. Be careful when dealing with infinity.

Summation Notation

In this section we need to do a brief review of summation notation or sigma notation. We'll start out with a list of numbers denoted as follows,

$$a_n,\ a_{n+1},\ a_{n+2},\ \ldots,\ a_{m-2},\ a_{m-1},\ a_m \qquad n<m$$

We want to add them up, in other words we want,

$$a_n + a_{n+1} + a_{n+2} + \ldots + a_{m-2} + a_{m-1} + a_m$$

For large lists this can be a fairly cumbersome notation so we introduce summation notation to denote these kinds of sums. The case above is denoted as follows.

$$\sum_{i=n}^{m} a_i = a_n + a_{n+1} + a_{n+2} + \ldots + a_{m-2} + a_{m-1} + a_m$$

The i is called the index of summation. This notation tells us to add all the a_i's up for all integers starting at n and ending at m.

For instance,

$$\sum_{i=0}^{4} \frac{i}{i+1} = \frac{0}{0+1} + \frac{1}{1+1} + \frac{2}{2+1} + \frac{3}{3+1} + \frac{4}{4+1} = \frac{163}{60} = 2.7166\overline{6}$$

$$\sum_{i=4}^{6} 2^i x^{2i+1} = 2^4 x^9 + 2^5 x^{11} + 2^6 x^{13} = 16x^9 + 32x^{11} + 64x^{13}$$

$$\sum_{i=1}^{4} f\left(x_i^*\right) = f\left(x_1^*\right) + f\left(x_2^*\right) + f\left(x_3^*\right) + f\left(x_4^*\right)$$

Properties

Here are a couple of formulas for summation notation. Note that they all start at *n=1*, but could in reality start anywhere.

1. $\displaystyle\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$ where c is any number. So, we can factor constants out of a summation.
2. $\displaystyle\sum_{i=1}^{n} \left(a_i \pm b_i\right) = \sum_{n=1}^{n} a_i \pm \sum_{n=1}^{n} b_i$ So we can break up a summation across a sum or difference.

Formulas

Here are a couple of nice formulas that we will find useful in a couple of sections.

1. $\displaystyle\sum_{i=1}^{n} c = cn$
2. $\displaystyle\sum_{i=1}^{n} i = \frac{n\left(n+1\right)}{2}$

3. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

4. $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

Example 1 Using the formulas and properties from above determine the value of the following summation.

$$\sum_{i=1}^{100} (3-2i)^2$$

Solution

Without the formulas and properties this could be computed by hand, but it would definitely take some time. The first thing that we need to do is square out the stuff being summed and then break up the summation using the properties.

$$\begin{aligned}\sum_{i=1}^{100} (3-2i)^2 &= \sum_{i=1}^{100} 9 - 12i + 4i^2 \\ &= \sum_{i=1}^{100} 9 - \sum_{i=1}^{100} 12i + \sum_{i=1}^{100} 4i^2 \\ &= \sum_{i=1}^{100} 9 - 12\sum_{i=1}^{100} i + 4\sum_{i=1}^{100} i^2\end{aligned}$$

Now, using the formulas, this is easy to compute,

$$\begin{aligned}\sum_{i=1}^{100} (3-2i)^2 &= 9(100) - 12\left(\frac{100(101)}{2}\right) + 4\left(\frac{100(101)(201)}{6}\right) \\ &= 1293700\end{aligned}$$

Constants of Integration

In this section we need to address a couple of topics about the constant of integration. Throughout most calculus classes we play pretty fast and loose with it and because of that many students don't really understand it or how it can be important.

First, let's address how we play fast and loose with it. Recall that technically when we integrate a sum or difference we are actually doing multiple integrals. For instance,

$$\int 15x^4 - 9x^{-2}\,dx = \int 15x^4\,dx - \int 9x^{-2}\,dx$$

Upon evaluating each of these integrals we should get a constant of integration for each integral since we really are doing two integrals.

$$\begin{aligned}\int 15x^4 - 9x^{-2}\,dx &= \int 15x^4\,dx - \int 9x^{-2}\,dx \\ &= 3x^5 + c + 9x^{-1} + k \\ &= 3x^5 + 9x^{-1} + c + k\end{aligned}$$

Since there is no reason to think that the constants of integration will be the same from each integral we use different constants for each integral.

Now, both c and k are unknown constants and so the sum of two unknown constants is just an unknown constant and we acknowledge that by simply writing the sum as a c.

So, the integral is then,

$$\int 15x^4 - 9x^{-2}\,dx = 3x^5 + 9x^{-1} + c$$

We also tend to play fast and loose with constants of integration in some substitution rule problems. Consider the following problem,

$$\int \cos\left(1+2x\right) + \sin\left(1+2x\right)dx = \frac{1}{2}\int \cos u + \sin u\,du \qquad\qquad u = 1+2x$$

Technically when we integrate we should get,

$$\int \cos\left(1+2x\right) + \sin\left(1+2x\right)dx = \frac{1}{2}\left(\sin u - \cos u + c\right)$$

Since the whole integral is multiplied by $\frac{1}{2}$, the whole answer, including the constant of integration, should be multiplied by $\frac{1}{2}$. Upon multiplying the $\frac{1}{2}$ through the answer we get,

$$\int \cos\left(1+2x\right) + \sin\left(1+2x\right)dx = \frac{1}{2}\sin u - \frac{1}{2}\cos u + \frac{c}{2}$$

However, since the constant of integration is an unknown constant dividing it by 2 isn't going to change that fact so we tend to just write the fraction as a c.

$$\int \cos\left(1+2x\right) + \sin\left(1+2x\right)dx = \frac{1}{2}\sin u - \frac{1}{2}\cos u + c$$

In general, we don't really need to worry about how we've played fast and loose with the constant of integration in either of the two examples above.

The real problem however is that because we play fast and loose with these constants of integration most students don't really have a good grasp on them and don't understand that there are times where the constants of integration are important and that we need to be careful with them.

To see how a lack of understanding about the constant of integration can cause problems consider the following integral.

$$\int \frac{1}{2x}\,dx$$

This is a really simple integral. However, there are two ways to integrate it and that is where the problem arises.

The first integration method is to just break up the fraction and do the integral.

$$\int \frac{1}{2x}\,dx = \int \frac{1}{2}\frac{1}{x}\,dx = \frac{1}{2}\ln|x| + c$$

The second way is to use the following substitution.

$$u = 2x \qquad du = 2dx \qquad \Rightarrow \qquad dx = \frac{1}{2}du$$

$$\int \frac{1}{2x}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + c = \frac{1}{2}\ln|2x| + c$$

Can you see the problem? We integrated exactly the same function and got very different answers. This doesn't make any sense. Integrating the same function should give us the same answer. We only used different methods to do the integral and both are perfectly legitimate integration methods. So, how can using different methods produce different answer?

The first thing that we should notice is that because we used a different method for each there is no reason to think that the constant of integration will in fact be the same number and so we really should use different letters for each.

More appropriate answers would be,

$$\int \frac{1}{2x}\,dx = \frac{1}{2}\ln|x| + c$$

$$\int \frac{1}{2x}\,dx = \frac{1}{2}\ln|2x| + k$$

Now, let's take another look at the second answer. Using a property of logarithms we can write the answer to the second integral as,

$$\begin{aligned}\int \frac{1}{2x}\,dx &= \frac{1}{2}\ln|2x| + k \\ &= \frac{1}{2}\left(\ln 2 + \ln|x|\right) + k \\ &= \frac{1}{2}\ln|x| + \frac{1}{2}\ln 2 + k\end{aligned}$$

Upon doing this we can see that the answers really aren't that different after all. In fact they only differ by a constant and we can even find a relationship between c and k. It looks like,

$$c = \frac{1}{2}\ln 2 + k$$

So, without a proper understanding of the constant of integration, in particular using different integration techniques on the same integral will likely produce a different constant of integration, we might never figure out why we got "different" answers for the integral.

Note as well that getting answers that differ by a constant doesn't violate any principles of calculus. In fact, we've actually seen a fact that suggested that this might happen. We saw a fact in the Mean Value Theorem section that said that if $f'(x) = g'(x)$ then $f(x) = g(x) + c$. In other words, if two functions have the same derivative then they can differ by no more than a constant.

This is exactly what we've got here. The two functions.

$$f(x) = \frac{1}{2}\ln|x|$$

$$g(x) = \frac{1}{2}\ln|2x|$$

have exactly the same derivative,

$$\frac{1}{2x}$$

and as we've shown they really only differ by a constant.

There is another integral that also exhibits this behavior. Consider,

$$\int \sin(x)\cos(x)\,dx$$

There are actually three different methods for doing this integral.

Method 1 :
This method uses a trig formula,

$$\sin(2x) = 2\sin(x)\cos(x)$$

Using this formula (and a quick substitution) the integral becomes,

$$\int \sin(x)\cos(x)\,dx = \frac{1}{2}\int \sin(2x)\,dx = -\frac{1}{4}\cos(2x) + c_1$$

Method 2 :
This method uses the substitution,

$$u = \cos(x) \qquad\qquad du = -\sin(x)\,dx$$

$$\int \sin(x)\cos(x)\,dx = -\int u\,du = -\frac{1}{2}u^2 + c_2 = -\frac{1}{2}\cos^2(x) + c_2$$

Method 3 :
Here is another substitution that could be done here as well.

$$u = \sin(x) \qquad\qquad du = \cos(x)\,dx$$

$$\int \sin(x)\cos(x)\,dx = \int u\,du = \frac{1}{2}u^2 + c_3 = \frac{1}{2}\sin^2(x) + c_3$$

So, we've got three different answers each with a different constant of integration. However, according to the fact above these three answers should only differ by a constant since they all have the same derivative.

In fact they do only differ by a constant. We'll need the following trig formulas to prove this.

$$\cos(2x) = \cos^2(x) - \sin^2(x) \qquad\qquad \cos^2(x) + \sin^2(x) = 1$$

Start with the answer from the first method and use the double angle formula above.

$$-\frac{1}{4}\left(\cos^2(x) - \sin^2(x)\right) + c_1$$

Now, from the second identity above we have,

$$\sin^2(x) = 1 - \cos^2(x)$$

so, plug this in,

$$\begin{aligned} -\frac{1}{4}\left(\cos^2(x) - \left(1 - \cos^2(x)\right)\right) + c_1 &= -\frac{1}{4}\left(2\cos^2(x) - 1\right) + c_1 \\ &= -\frac{1}{2}\cos^2(x) + \frac{1}{4} + c_1 \end{aligned}$$

This is then answer we got from the second method with a slightly different constant. In other words,

$$c_2 = \frac{1}{4} + c_1$$

We can do a similar manipulation to get the answer from the third method. Again, starting with the answer from the first method use the double angle formula and then substitute in for the cosine instead of the sine using,

$$\cos^2(x) = 1 - \sin^2(x)$$

Doing this gives,

$$\begin{aligned} -\frac{1}{4}\left(\left(1 - \sin^2(x)\right) - \sin^2(x)\right) + c_1 &= -\frac{1}{4}\left(1 - 2\sin^2(x)\right) + c_1 \\ &= \frac{1}{2}\sin^2(x) - \frac{1}{4} + c_1 \end{aligned}$$

which is the answer from the third method with a different constant and again we can relate the two constants by,

$$c_3 = -\frac{1}{4} + c_1$$

Summary

So, what have we learned here? Hopefully we've seen that constants of integration are important and we can't forget about them.

We often don't work with them in a Calculus I course, yet without a good understanding of them we would be hard pressed to understand how different integration methods and apparently produce different answers.

Area and Volume Formulas

In this section we will derive the formulas used to get the area between two curves and the volume of a solid of revolution.

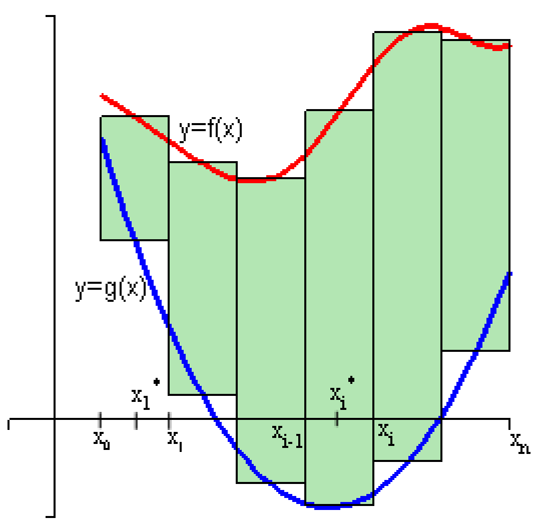
Area Between Two Curves

We will start with the formula for determining the area between *y=f(x)* and *y=g(x)* on the interval [*a*,*b*].

We will start by assuming that $f(x) \geq g(x)$ on [*a*,*b*]. We will now proceed much as we did when we looked that the Area Problem in the Integrals Chapter. We will first divide up the interval into *n* equal subintervals each with length,

$$\Delta x = \frac{b-a}{n}$$

Next, pick a point in each subinterval, x_i^*, and we can then use rectangles on each interval as follows.



The height of each of these rectangles is given by,

$$f\left(x_i^*\right) - g\left(x_i^*\right)$$

and the area of each rectangle is then,

$$\left(f\left(x_i^*\right) - g\left(x_i^*\right)\right)\Delta x$$

So, the area between the two curves is then approximated by,

$$A \approx \sum_{i=1}^{n}\left(f\left(x_i^*\right) - g\left(x_i^*\right)\right)\Delta x$$

The exact area is,

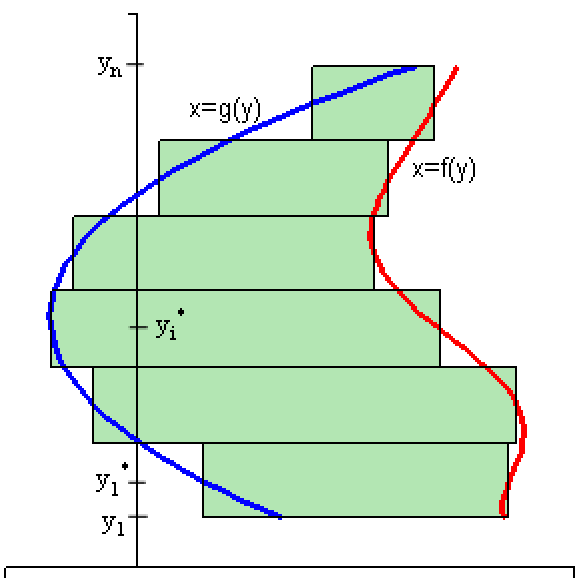
$$A = \lim_{n\to\infty}\sum_{i=1}^{n}\left(f\left(x_i^*\right) - g\left(x_i^*\right)\right)\Delta x$$

Now, recalling the definition of the definite integral this is nothing more than,

$$A = \int_a^b f\left(x\right) - g\left(x\right)dx$$

The formula will work provided the two functions are in the form $y=f(x)$ and $y=g(x)$. However, not all functions are in that form. Sometimes we will be forced to work with functions in the form between $x=f(y)$ and $x=g(y)$ on the interval $[c,d]$ (an interval of y values…).

When this happens the derivation is identical. First we will start by assuming that $f\left(y\right) \geq g\left(y\right)$ on $[c,d]$. We can then divide up the interval into equal subintervals and build rectangles on each of these intervals. Here is a sketch of this situation.



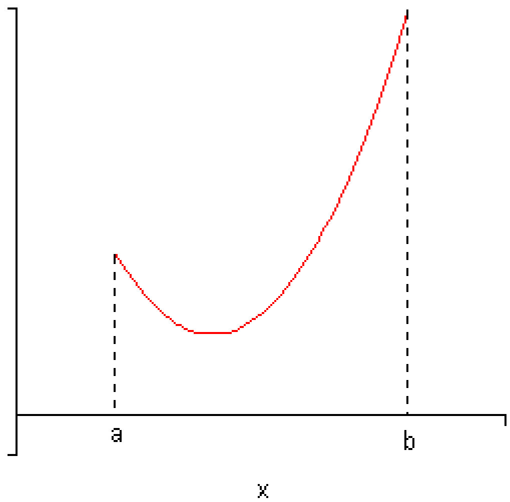
Following the work from above, we will arrive at the following for the area,

$$A = \int_c^d f(y) - g(y)\,dy$$

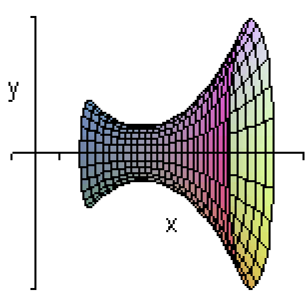
So, regardless of the form that the functions are in we use basically the same formula.

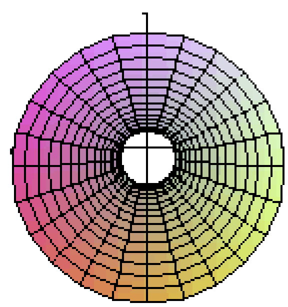
Volumes for Solid of Revolution

Before deriving the formula for this we should probably first define just what a solid of revolution is. To get a solid of revolution we start out with a function, *y=f(x)*, on an interval [*a*,*b*].



We then rotate this curve about a given axis to get the surface of the solid of revolution. For purposes of this derivation let's rotate the curve about the *x*-axis. Doing this gives the following three dimensional region.





The image on the right shows the 3-D object from the side and the image on the left shows the 3-D object from the end. Note that we didn't cap the ends here and are really just showing the surface of the solid of revolution.

What we want to determine is the volume of the interior of the object. To do this we will proceed much as we did for the area between two curves case. We will first divide up the interval into *n* subintervals of width,

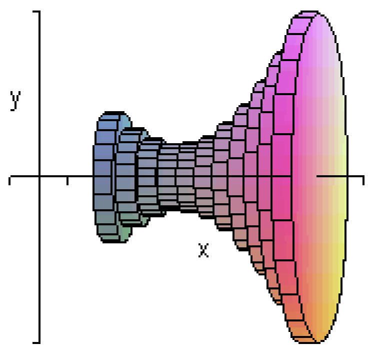
$$\Delta x = \frac{b-a}{n}$$

We will then choose a point from each subinterval, x_i^*.

Now, in the area between two curves case we approximated the area using rectangles on each subinterval. For volumes we will use disks on each subinterval to approximate the area. The area of the face of each disk is given by $A\left(x_i^*\right)$ and the volume of each disk is

$$V_i = A\left(x_i^*\right)\Delta x$$

Here is a sketch of this,



The volume of the region can then be approximated by,

$$V \approx \sum_{i=1}^{n} A\left(x_i^*\right)\Delta x$$

The exact volume is then,

$$\begin{aligned} V &= \lim_{n\to\infty}\sum_{i=1}^{n} A\left(x_i^*\right)\Delta x \\ &= \int_a^b A\left(x\right)dx \end{aligned}$$

So, in this case the volume will be the integral of the cross-sectional area at any *x*, *A(x)*. Note as well that, in this case, the cross-sectional area is a circle and we could go farther and get a formula for that as well. However, the formula above is more general and will work for any way of getting a cross section so we will leave it like it is.

In the sections where we actually use this formula we will also see that there area ways of generating the cross section that will actually give a cross-sectional area that is a function of *y* instead of *x*. In these cases the formula will be,

$$V = \int_c^d A(y)\,dy, \qquad c \le y \le d$$

In this case we looked at rotating a curve about the *x*-axis, however, we could have just as easily rotated the curve about the *y*-axis. In fact we could rotate the curve about any vertical or horizontal axis and in all of these, case we can use one or both of the following formulas.

$$V = \int_a^b A(x)\,dx \qquad\qquad V = \int_c^d A(y)\,dy$$