THE UNIVERSITY OF AKRON
Department of Theoretical and Applied Mathematics

LESSON 5: GRAPHS OF OTHER TRIGONOMETRIC FUNCTIONS

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1. Introduction

In the previous lesson we constructed the graphs of the basic sine and cosine functions, and developed techniques for graphing more general waves related to these functions. The current lesson is devoted to examining graphs of the remaining trigonometric functions: the tangent, cotangent, secant, and cosecant.

The next two sections describe the graphs of these functions as well as presenting strategies for graphing modified (co)tangent and (co)secant functions. These sections contain information and examples that require a thorough understanding of the terms period[1] and phase shift (or translation), and their effect on the graphs of the sine and cosine functions. The reader may wish to review these concepts as presented in Lesson 4 before proceeding.

[1]The (co)tangent and (co)secant functions do not possess the property referred to as amplitude. However, premultiplication of these functions by constants does affect their values and graphs. This effect is discussed in this lesson.

2. Graphs of tangent and cotangent waves

We begin with the graph of the tangent function. Recall that if (x, y) is the point on the unit circle determined by an angle of radian measure t, then $\tan t = y/x$ provided that $x \neq 0$ (as is the case when $t = \pi/2$). Table 5.1 contains values for the tangent function at some special angles. (Also see Table 2.2 in Lesson 2.) Notice that the tangent function increases from 0 to ∞ as t increases on the interval $[0, \pi/2)$. Carefully plotting the points in the table produces the graph given in Figure 5.1.

The notation $\tan t \to \infty$ as $t \longrightarrow \pi/2$ with $t < \pi/2$ in the last column of Table 5.1 means that the tangent function approaches ∞, or increases without bound, as t approaches $\pi/2$ from the left. Graphically, this means that $y(t) = \tan t$ has a **vertical asymptote** at $t = \pi/2$.

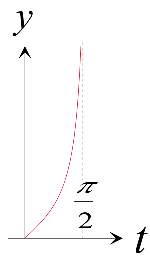


Figure 5.1: Graph of the tangent function on $[0, \pi/2)$.

Radians	0	$\pi/6$	$\pi/4$	$\pi/3$	$t \longrightarrow \pi/2$ with $t < \pi/2$
$\tan t$	0	$\sqrt{3}/3$	1	$\sqrt{3}$	$\tan t \longrightarrow \infty$

Table 5.1: Values of the tangent function on $[0, \pi/2)$.

Exercise 5 establishes that the tangent function satisfies

$$\tan(-t) = -\tan t.$$

Such functions are called **odd**[2] because their graphs reflect about the origin. That is, if a point (a, b) is on the graph of an odd function, then so is the point $(-a, -b)$. Such functions are often said to be **symmetric with respect to the origin**. Hence, the curve in Figure 5.1 can be reflected about the origin to obtain the graph of the tangent function on $(-\pi/2, \pi/2)$ depicted in Figure 5.2. Observe that the curve decreases from 0 to $-\infty$ as t approaches $-\pi/2$ from the right, demonstrating its asymptotic symmetry about the origin. As with the graphs of the sine and cosine functions, the reader should commit this graph to memory.

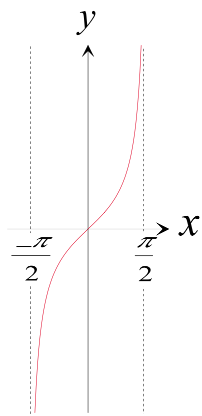


Figure 5.2: Graph of the tangent function on $(-\pi/2, \pi/2)$.

[2]Likewise, a function $f(t)$ is **even** if $f(-t) = -f(t)$ for all t in the domain of f. This means that the graph of f is symmetric with respect to the y-axis.

It was shown in Exercise 5 of Lesson 2 that the tangent function has period π. Hence, the complete graph of this function is easily obtained by duplicating the curve in Figure 5.2 on contiguous intervals of length π, beginning with the interval $(-\pi/2, \pi/2)$. Several such waves of the tangent function are given in Figure 5.3. Observe that the vertical asymptotes occur at $t = \dfrac{(2k+1)\pi}{2}$ where k is any integer.

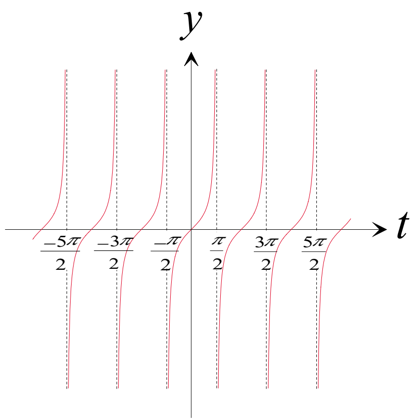
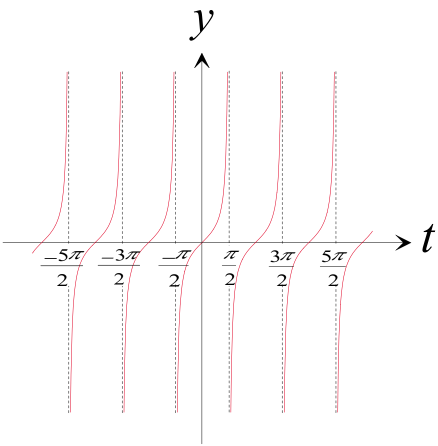


Figure 5.3: Graph of the tangent function.

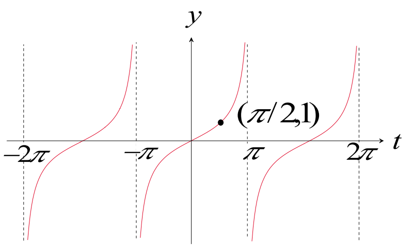
As with the sine and cosine functions, the tangent function can be modified, for example, by changing its period. The function $y(t) = \tan(at)$ has period $\frac{\pi}{|a|}$ since $y\left(t + \frac{\pi}{a}\right) = y(t)$ for all t in its domain. The following example illustrates this type of modification to the tangent function. While examining the graph in this (and the next) example, readers should refer to the graph given in Figure 5.3 (reproduced below for convenience) of the basic tangent function in an effort to understand why the changes in the appearance of the curves occur.



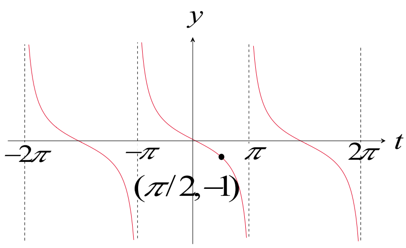
Example 1 *Sketch a graph of the function* $y\,(t) = \tan\left(\frac{t}{2}\right)$

Solution: *The factor of* $\frac{1}{2}$ *means that* $y\,(t)$ *has a period of* $2\pi = \frac{\pi}{1/2}$*. Consequently, the graph of a wave of this function is similar to that for the tangent function given in Figure 5.3 except the vertical asymptotes are* 2π *units apart. These asymptotes occur at* $t = (2k+1)\pi$ *where* k *is any integer, or twice as far apart as those for the basic tangent function. The graph of three waves of* $y\,(t)$ *appears in Figure (a).*

Example 2 *Let* $y\,(t) = \tan\left(-\frac{t}{2}\right)$*. Since the tangent function is odd,* $y\,(t)$ *can be written as* $y\,(t) = -\tan\left(\frac{t}{2}\right)$*. Hence, the graph of* $y\,(t)$ *is a reflection of that for Example 1 about the* t*-axis. Three waves of its graph is given in Figure (b). Compare this graph with the one in Figure (a) and note the symmetry between the two.*



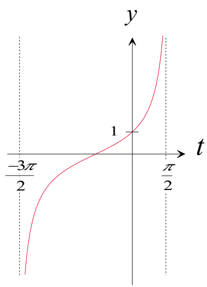
(a) $y(t) = \tan\left(\frac{t}{2}\right)$



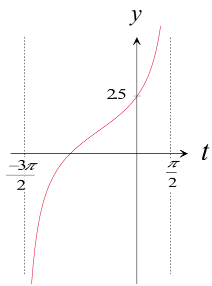
(b) $y(t) = \tan\left(-\frac{t}{2}\right)$

Example 3 *Graph one wave of* $y_1(t) = \tan\left(\frac{t}{2} + \frac{\pi}{4}\right)$ *and* $y_2(t) = \frac{3}{2}\tan\left(\frac{t}{2} + \frac{\pi}{4}\right) + 1$.

Note that both functions have period 2π*. The graph of* $y_1(t)$*, pictured in Figure (a), is essentially a shift of the function graphed in Example 1 to the left by* $\frac{\pi}{2}$ *units (since* $\frac{t}{2} + \frac{\pi}{4} = \frac{1}{2}(t + \frac{\pi}{2})$*). Multiplying* $y_1(t)$ *by* $\frac{3}{2}$ *stretches its graph vertically by* $\frac{3}{2}$ *units. We can then produce the graph of* $y_2(t)$ *in Figure (b) by raising the graph of* $\frac{3}{2}y_1(t)$ *one unit. Note that* $y_1(t)$ *crosses the* y*-axis at* $1 = y_1(0)$ *while* $y_2(t)$ *crosses this axis at* $2.5 = y_2(0)$*. If desired, additional waves of these curves can be obtained by exploiting their periodic behavior.*



(a) $y_1(t) = \tan\left(\frac{t}{2} + \frac{\pi}{4}\right)$



(b) $y_2(t) = \frac{3}{2}\tan\left(\frac{t}{2} + \frac{\pi}{4}\right) + 1$

It is possible to generate the graph of the cotangent function by applying techniques similar to those utilized above for the tangent function. Table 5.2 gives values of the cotangent function for some special angles. Note that this function approaches ∞ as t approaches 0 from the right and it approaches $-\infty$ as t approaches π from the left. Table 5.2 and the fact that the cotangent has period π can be used to produce the graph given in Figure 5.4. The sketch leads us to believe that the cotangent function is odd. This is addressed in Exercise 5.

The reader should note the similarities and differences between the graphs of the basic tangent and cotangent functions. (See Exercise 3.)

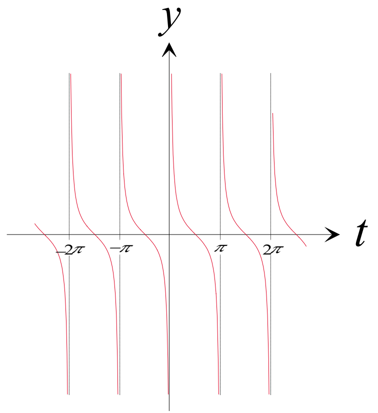


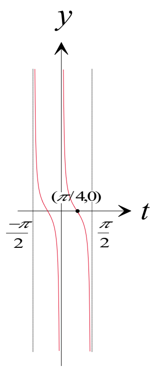
Figure 5.4: $y(t) = \cot t$

Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
cot	∞	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0	$-\frac{\sqrt{3}}{3}$	-1	$-\sqrt{3}$	$-\infty$

Table 5.2: Values of the cotangent function on $(0, \pi)$.

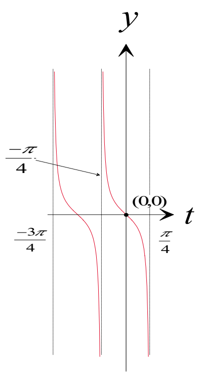
Example 4 *Graph two waves of the functions*
(a) $y_1(t) = \cot(2t)$, *(b)* $y_2(t) = \cot\left(2t - \frac{\pi}{2}\right)$, *and (c)* $y_3(t) = -\frac{3}{2}\cot\left(2t - \frac{\pi}{2}\right) - 1$.

Solution: *The factor of* 2 *in the argument of* $y_1(t)$ *causes it to have period* $\pi/2$. *Consequently, the graph of this function is similar to that in Figure 5.4 except the vertical asymptotes would be* $\frac{\pi}{2}$ *units apart. A sketch of two waves of* $y_1(t)$ *is given below. (This example is continued on the next page.)*

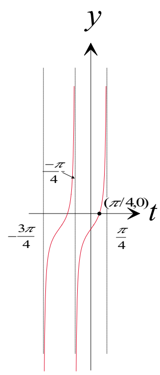


(a) $y_1(t) = \cot(2t)$

(Example 4 continued.) The graph of $y_2(t) = \cot\left(2t - \frac{\pi}{2}\right)$ *is essentially a shift of* $y_1(t)$ *to the right by a factor of* $\frac{\pi}{4}$*. A sketch of* $y_2(t)$ *is given in Figure (b). Multiplying* $y_2(t)$ *by* $-3/2$ *first stretches its graph vertically by* $3/2$ *units and then reflects the result about the t-axis. Lowering the resulting curve by one unit produces the graph of* $y_3(t)$*, which is given in Figure (c) below.*



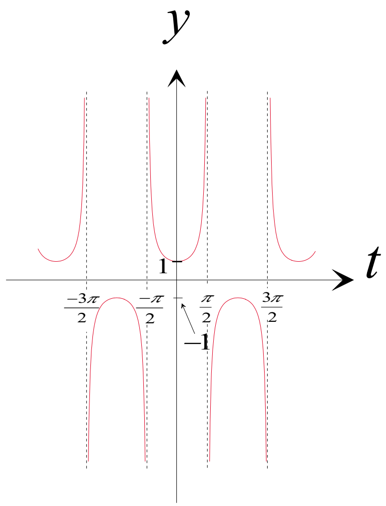
(b) $y_2(t) = \cot(2t - \frac{\pi}{2})$



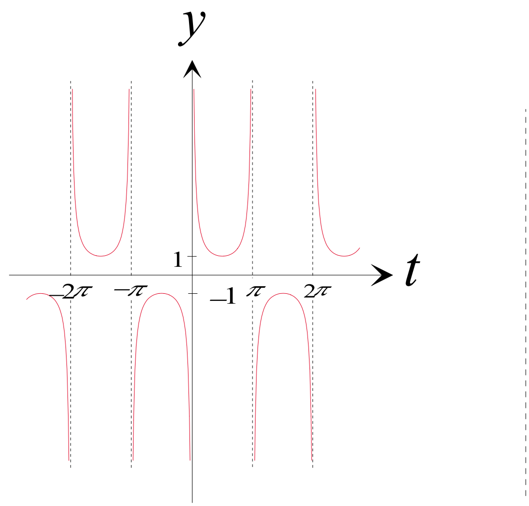
(c) $y_3 = -\frac{3}{2}\cot(2t - \frac{\pi}{2}) - 1$

3. Graphs of secant and cosecant

The graphs of the secant and cosecant functions appear below. These curves are easily obtained by plotting points and exploiting their 2π periodic behavior. Note that the distance between the vertical asymptotes of these functions is π, or half their period. The graphs correctly suggest that the secant function is even and the cosecant function is odd. (See Exercise 5.)



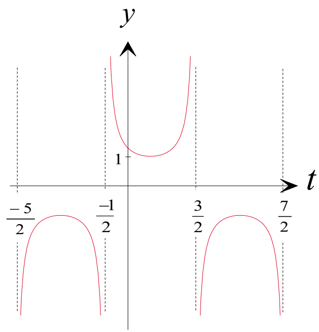
(a) $y = \sec x$



(b) $y = \csc t$

Example 5 *Sketch a portion of the graph of* $y(t) = \sec\left(\frac{\pi}{2}t - \frac{\pi}{4}\right)$.

Solution: *The period of* $y(t)$ *is* $4 = \frac{2\pi}{\pi/2}$ *so the distance between its vertical asymptotes is* 2 *units or half its period. The graph is translated or shifted to the right by a factor of* $\frac{1}{2}$ *(because* $\left(\frac{\pi}{2}t - \frac{\pi}{4}\right) = \frac{\pi}{2}\left(t - \frac{1}{2}\right)$*) so the asymptotes occur at* $t = -1/2 + 2k$ *where* k *is any integer. We can use this information to produce the portion of the graph of* $y(t)$ *given in the figure below.*



$$y = \sec\left(\frac{\pi}{2}t - \frac{\pi}{4}\right)$$

4. Exercises

EXERCISE 1. Graph one wave of each of the following functions:

(a) $y_1(t) = \tan\left(\frac{\pi}{2}t\right)$
(b) $y_2(t) = \tan\left(\frac{\pi}{2}t + \frac{\pi}{4}\right)$
(c) $y_3(t) = -\frac{1}{2}\tan\left(\frac{\pi}{2}t + \frac{\pi}{4}\right) - 1.$

EXERCISE 2. Graph one wave of each of the following functions:

(a) $y_1(t) = \cot\left(\frac{\pi}{2}t\right)$
(b) $y_2(t) = \cot\left(\frac{\pi}{2}t + \frac{\pi}{4}\right)$
(c) $y_3(t) = -\frac{1}{2}\cot\left(\frac{\pi}{2}t + \frac{\pi}{4}\right) - 1$

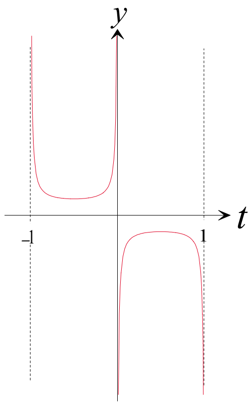
EXERCISE 3. Verify graphically that $\cot t = -\tan\left(t - \frac{\pi}{2}\right)$.

EXERCISE 4. Sketch a graph of $y(t) = -\frac{1}{2}\csc\left(\frac{\pi}{2}t + \frac{\pi}{4}\right)$.

EXERCISE 5. Use the definitions and other properties of the four trigonometric functions discussed in this lesson to verify their even or odd behavior.

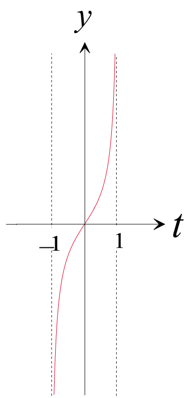
The next exercise requires that we find a trigonometric function whose graph is given. There are, of course, infinitely many solutions to this problem. Although an infinity of functions that graph the given curve are presented in the solution, the problem requires only one.

EXERCISE 6. One wave of the graph of a trigonometric function $y(t)$ appears in the graph below. Find a formula for the function if $y\left(\frac{1}{6}\right) = \frac{1}{2}$.


y
t
–1
1

Solutions to Exercises

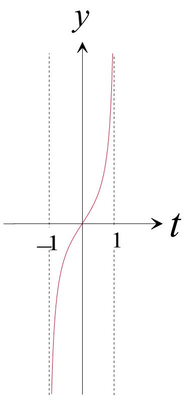
Exercise 1(a) Graph one wave of $y_1(t) = \tan\left(\frac{\pi}{2}t\right)$.
Solution: The period of this function is $2 = \frac{\pi}{\pi/2}$. Consequently, the graph of a wave of $y_1(t)$, given in Figure (a), is essentially the same as the basic tangent function with asymptotes located at odd integers.



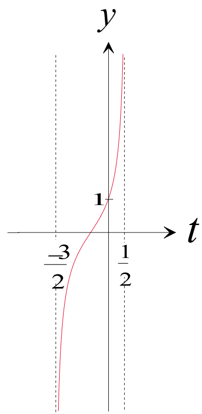
(a) $y_1(t) = \tan\left(\frac{\pi}{2}t\right)$

□

Exercise 1(b) Graph one wave of $y_2(t) = \tan\left(\frac{\pi}{2}t + \frac{\pi}{4}\right)$.
Solution: The period of this functions is 2 and it has a shift of $\frac{1}{2}$ unit to the left. Figure (b) is a graph of a wave of $y_2(t)$. It can be obtained by shifting the graph of $y_1(t)$ in Figure (a) $\frac{1}{2}$ unit to the left.



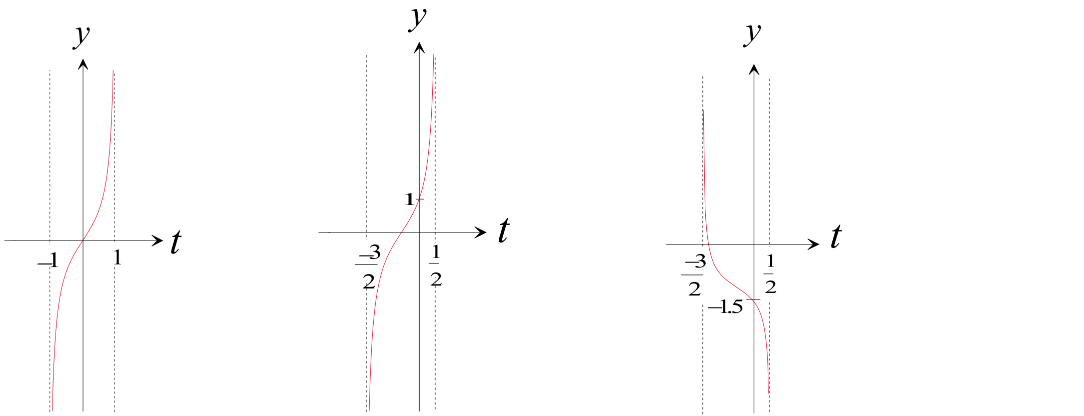
(a) $y_1(t) = \tan\left(\frac{\pi}{2}t\right)$



(b) $y_2(t) = \tan\left(\frac{\pi}{2}t + \frac{\pi}{4}\right)$

$\square$

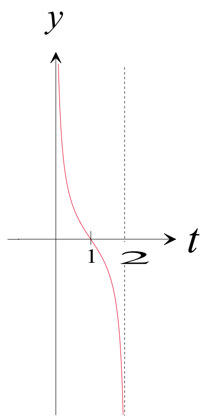
Exercise 1(c) Graph one wave of $y_3(t) = -\frac{1}{2}\tan\left(\frac{\pi}{2}t + \frac{\pi}{4}\right) - 1.$
Solution: To graph $y_3(t)$ first reflect the graph of r $y_2(t)$ in Figure (b) about the t-axis and then compress the resulting graph. This accounts for the factor of $-1/2$ in $y_3(t)$. Now lower this last curve by one unit because of the factor -1. The desired graph is given in Figure (c). Note that $y_3(0) = -1.5$.



(a) $y_1(t) = \tan\left(\frac{\pi}{2}t\right)$ (b) $y_2(t) = \tan\left(\frac{\pi}{2}t + \frac{\pi}{4}\right)$ (c) $y_3(t) = -\frac{1}{2}\tan(\frac{\pi}{2}t + \frac{\pi}{4}) - 1$

$\square$

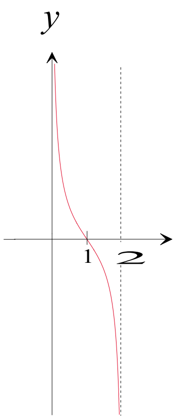
Exercise 2(a) Graph one wave of $y_1(t) = \cot\left(\frac{\pi}{2}t\right)$.
Solution: Observe that the period of $y_1(t)$ is 2 so is graph its similar to that for the basic cotangent function except the asymptotes are located at the even integers. Figure (a) traces one wave of $y_1(t)$.



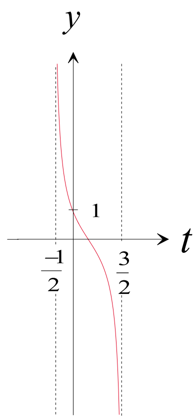
(a) $y_1(t) = \cot\left(\frac{\pi}{2}t\right)$

□

Exercise 2(b) Graph one wave of $y_2(t) = \cot\left(\frac{\pi}{2}t + \frac{\pi}{4}\right)$.
Solution: The period of this function is 2 and it has a shift of $\frac{1}{2}$ unit to the left. Figure (b) is a graph of a wave of $y_2(t)$. It can be obtained by shifting the graph of $y_1(t)$ in Figure (a) $\frac{1}{2}$ unit to the left.



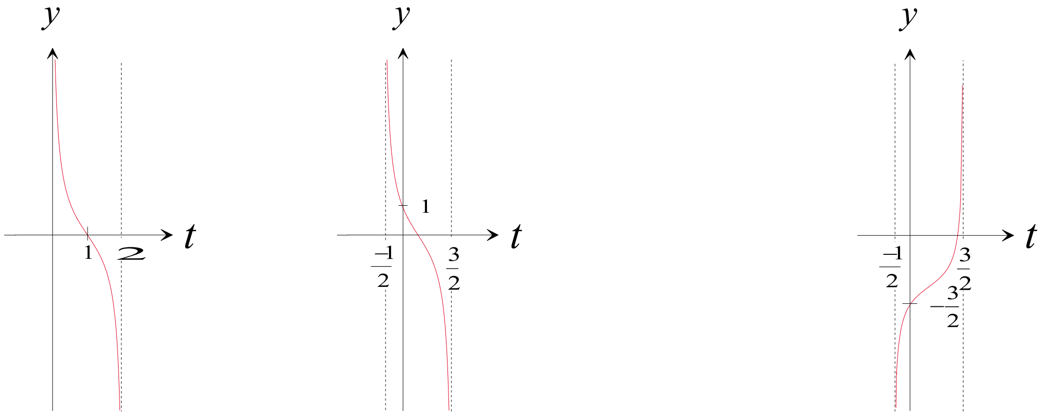
(a) $y_1(t) = \cot\left(\frac{\pi}{2}t\right)$



(b) $y_2(t) = \cot\left(\frac{\pi}{2}t + \frac{\pi}{4}\right)$

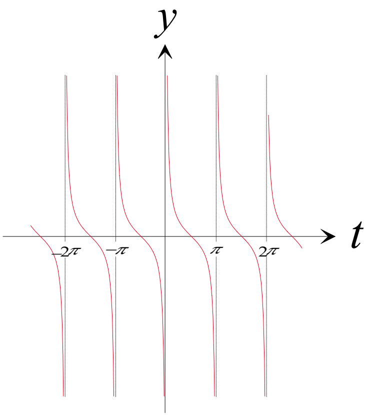
$\square$

Exercise 2(c) Graph one wave of $y_3(t) = -\frac{1}{2}\cot\left(\frac{\pi}{2}t + \frac{\pi}{4}\right) - 1$.
Solution: To graph $y_3(t)$ first reflect the graph of $y_2(t)$ in Figure (b) about the t-axis and then compress the resulting graph. This accounts for the factor of $-1/2$ in $y_3(t)$. Now lower this last curve by one unit because of the factor -1. The desired graph is given in Figure (c). Note that $y_3(0) = -1.5$.



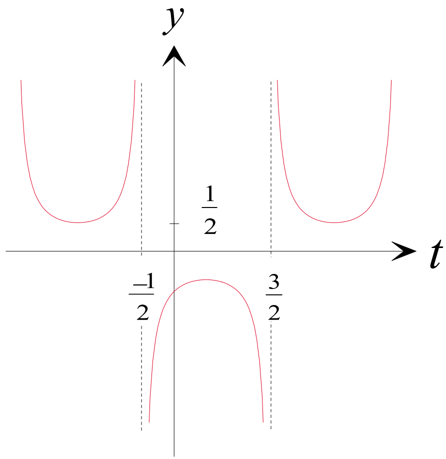
(a) $y_1(t) = \cot\left(\frac{\pi}{2}t\right)$ (b) $y_2(t) = \cot\left(\frac{\pi}{2}t + \frac{\pi}{4}\right)$ (c) $y_3(t) = -\frac{1}{2}\cot\left(\frac{\pi}{2}t + \frac{\pi}{4}\right) - 1$

Exercise 3. Begin with the function $y(t) = -\tan\left(t - \frac{\pi}{2}\right)$ which has period π. The graph of $y(t)$ can be obtained from that of the basic tangent function by shifting it $\frac{\pi}{2}$ units to the left and then reflecting the result about the t-axis. The graph of this curve, given below, is the same as that for the cotangent function given in Figure 5.4.



Exercise 3

Exercise 4. Sketch a graph of $y(t) = -\frac{1}{2}\csc\left(\frac{\pi}{2}t + \frac{\pi}{4}\right)$.
Solution: The graph of $y(t)$ can be obtained by first noting that its period is 4. The factor of $\frac{\pi}{4}$ in the argument of $y(t)$ results in a shift to the left of $\frac{1}{2}$ unit because $\frac{\pi}{2}t + \frac{\pi}{4} = \frac{\pi}{2}\left(t + \frac{1}{2}\right)$. Finally, the factor of $-\frac{1}{2}$ in front of the cosecant causes a reflection about the t-axis as well as a scaling. The graph of $y(t)$ is given below.



Exercise 4

Exercise 5. In section 4 of Lesson 2 it was noted that if (x, y) is the point on the unit circle determined by an angle of radian measure t, then $(x, -y)$ corresponds to the point determined by the angle $(-t)$ radians. Consequently,

$$\tan(-t) = \frac{-y}{x} = -\frac{y}{x} = -\tan t$$

which verifies that the tangent function is odd. The cotangent function is also odd since

$$\cot(-t) = \frac{1}{\tan(-t)} = \frac{1}{-\tan t} = -\frac{1}{\tan t} = -\cot t.$$

The secant function is even because

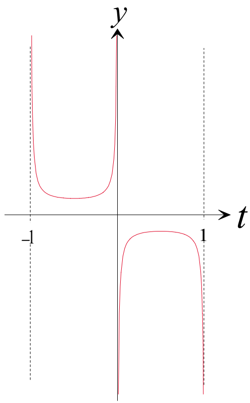
$$\sec(-t) = \frac{1}{x} = \sec t.$$

This also proves that the cosine function is even since $\cos t = 1/\sec t$. Finally, the cosecant function is odd because

$$\csc(-t) = \frac{1}{-y} = -\frac{1}{y} = -\csc t.$$

Note that this also means that the sine function is odd. Exercise 5

Exercise 6. One wave of the graph of a trigonometric function $y\,(t)$ appears in the graph below. Find a formula for the function if $y\left(\frac{1}{6}\right) = \frac{1}{2}$.



Solution: The sketched portion of the function has the shape of a secant or cosecant function and can be solved using either. We restrict our attention to the secant function. Consequently, we may assume that the curve has the form

$$y\,(t) = A\sec(at + b).$$

The curve completes one wave in 2 units suggesting the desired function has period 2. Since the period of the secant function is 2π we solve the equation $\frac{2\pi}{a} = 2$ for a.

Thus, $a = \pi$. so $y(t) = A\sec(\pi t + b)$ where A and b are to be determined. Next, we find b. The curve $\widetilde{y}(t) = \sec(\pi t)$ has a vertical asymptote at $t = -\frac{1}{2}$ while y has what appears to be a concomitant asymptote at $t = -1$. This suggests we solve $\pi(-\frac{1}{2}) = \pi(-1) + b$ for b, yielding $b = \frac{\pi}{2}$. Thus, $y(t) = A\sec(\pi t + \frac{\pi}{2})$. Finally, to determine A we note that we are given $\frac{1}{2} = y(-\frac{1}{6})$ so

$$\frac{1}{2} = y(-\frac{1}{6}) = A\sec(-\frac{\pi}{6} + \frac{\pi}{2}) = A\sec(\frac{\pi}{3}) = 2A.$$

It follows that $A = \frac{1}{4}$. Consequently, the curve is given by $y(t) = \frac{1}{4}\sec\left(\pi t + \frac{\pi}{2}\right)$.

The curve has an infinite number of representations using the secant function. In the above solution we used the vertical asymptote $t = -1$. We could have used any of the vertical asymptotes located at $t = (2k+1)$ where k denotes any integer, positive or negative.. (Why could we not use the asymptotes at $2k$ where k is an integer?. Hint: Use the asymptote $t = 0$, construct the resulting graph, and compare it with the given sketch.). If we use the asymptote $t = (2k+1)$, k an integer, we find the general solution by solving $\pi(-\frac{1}{2}) = \pi(2k+1) + b$ for b. This gives $b = -(2k+\frac{3}{2})\pi$ yielding the solution $y = \frac{1}{4}\sec(\pi t - (2k+\frac{3}{2})\pi) = \frac{1}{4}\sec(\pi t + \frac{3}{2}\pi - 2k)$ for any integer k. This simply reflects the periodic nature of the secant function.

A solution using the cosecant function is $y(t) = -\frac{1}{4}\csc(\pi t)$. To verify this analytically recall that $\csc(-t) = \csc t$ and $\csc t = \sec\left(\frac{\pi}{2} - t\right)$. Hence,

$$-\frac{1}{4}\csc(\pi t) = \frac{1}{4}\csc(-\pi t) = \frac{1}{4}\sec\left(\frac{\pi}{2} - (-\pi t)\right) = \frac{1}{4}\sec\left(\pi t + \frac{\pi}{2}\right).$$

Exercise 6